GEORGE SALTER ACADEMY MATHEMATICS

Year 12
A-Level Mathematics (Pure)

Mathematics Knowledge Organisers

Autumn Term 2024

Algebraic Expressions Cheat Sheet

In this chapter you are introduced to simple algebraic concepts that you may have come across before in your previous studies.

Index Laws

There are four key index laws that you need to know for not just this chapter but for the entirety of your maths course.

- $a^c \times a^d = a^{c+d}$
- $a^c \div a^d = a^{c-d}$
- $(a^c)^d = a^{cd}$
- $(ab)^c = a^c b^c$

Where a & b are the bases and c & d are the powers

Example 1: Simplifying expressions using index laws

a) $x^4 \times x^3 = x^{4+3} = x^7$

b) $\frac{4y^6}{2y^3} = \frac{4}{2} \times \frac{y^6}{y^3} = 2y^{6-3} = 2y^3$

c) $(z^2)^4 = z^{2\times 4} = z^8$

d) $(x^2y^3)^3 = x^{2\times 3}y^{3\times 3} = x^6y^9$

Negative and Fractional Indices

Indices (powers) can come in the form of fractions or negative numbers. The index laws can still be applied contingent on the powers being rational.

- $a^0 = 1$
- $a^{-c} = \frac{1}{a^c}$
- $a^{\frac{1}{c}} = \sqrt[c]{a}$
- $a^{\frac{c}{d}} = \sqrt[d]{a^c}$

Example 2: Simplify the following expressions

a) $x^{\frac{2}{3}} = \sqrt[3]{x^2}$

b) $36x^2 \div 6x^{-1} = 6x^{2-(-1)} = 6x^3$

c) $(81y^2)^{-\frac{1}{2}} = \frac{1}{(81y^2)^{\frac{1}{2}}} = \frac{1}{\sqrt{81}\times\sqrt{y^2}} = \frac{1}{9y}$

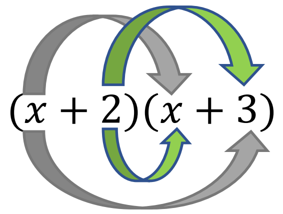
Example 3: Given that $s = t^2$, express each of the following in terms of t.

a) $s^{\frac{2}{3}} = (t^2)^{\frac{2}{3}} = t^{2\times\frac{2}{3}} = t^{\frac{4}{3}}$

b) $s^{-\frac{1}{4}} = \frac{1}{s^{\frac{1}{4}}} = \frac{1}{(t^2)^{\frac{1}{4}}} = \frac{1}{t^{2\times\frac{1}{4}}} = \frac{1}{t^{\frac{1}{2}}}$ *or* $\frac{1}{\sqrt{t}}$

Expanding Brackets

When expanding brackets for the product of two expressions, you have to multiply each term in the first expression by each term in the second expression and simplify the final product of this by collecting like terms.



$(x+2)$ is the first expression and $(x+3)$ is the second expression.
The first term in the first expression is x which is multiplied by the terms x and 3 in the second expression as indicated by the grey arrows.
The second term in the first expression is 2 which is also multiplied by the terms x and 3 in the second expression as indicated by the green arrows.

Therefore, $(x+2)(x+3) = x(x+3) + 2(x+3) = x^2 + 3x + 2x + 6$
Collecting the like terms, $(x+2)(x+3) = x^2 + 5x + 6$

Example 4: Expand the following brackets and simplify

a) $3(p+3)(p+2) = (3\times p + 3\times 3)(p+2) = (3p+9)(p+2)$
$= 3p(p+2) + 9(p+2)$
$= 3p^2 + 6p + 9p + 18$
$= 3p^2 + 15p + 18$

b) $(q+1)(q+2)(q+3)$

- *Start by expanding the first two brackets*
  $(q+1)(q+2) = q^2 + 2q + q + 2 = (q^2 + 3q + 2)$

- Rewrite the initial expression as $(q^2+3q+2)(q+3)$ and expand
  $(q^2+3q+2)(q+3)$
  $= q^2(q+3) + 3q(q+3) + 2(q+3)$
  $= q^3 + 3q^2 + 3q^2 + 9q + 2q + 6$
  $= q^3 + 6q^2 + 11q + 6$

Factorising

Factorising is the reverse of expanding brackets. When expanding brackets, you find the product of two or more expressions, however when you find the factors of a given expression it is called factorising.

$\div 4$

$$(4x + 24) = 4(x + 6)$$

The common factor of both terms in the expression is 4

To factorise quadratic expressions with the form $ax^2 + bx + c$ where a, b and c are areal numbers and $a \neq 0$.

- Calculate the product of $a \times c$ and find two factors of this product which add up to b.
- Rewrite the initial expression and substitute the bx term with the two factors found before.
- Factorise the first two terms and the last two terms of the rewritten expression.
- Simplify by taking out the common factor.

Example 5: Factorise the following expressions

a) $x^2 + 5x + 6$
$a = 1, b = 5, c = 6$

Two factors of $a \times c$ which also add up to b need to be calculated. Hence agree with the following statements:

1. $\_?\_ \times \_?\_ = a \times c = 1 \times 6 = 6$
2. $\_?\_ + \_?\_ = b = 5$

The two numbers which agree with both the statements are 3 and 2.

The 'b' term can now be rewritten using the two factors found and hence the expression will take the form of:

$$x^2 + 2x + 3x + 6$$

Now factorising the first two terms and the last two terms:

$x(x+2) + 3(x+2)$
$= (x+2)(x+3)$

- **Difference of two squares**: $x^2 - y^2 = (x+y)(x-y)$

Surds and Rationalising Denominators

Surds are irrational numbers which come in the exact form of $\sqrt{a}$ and where a is not a square number. The following rules can be applied to surds:

- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

You may come across fractions with the denominator being a surd. To get rid of this irrational number in the denominator we can rationalise it by using the following rules/methods which apply to different forms of fractions:

$\frac{1}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}}$ For this form we multiply the numerator and denominator by $\sqrt{a}$

$\frac{1}{a-\sqrt{b}} \times \frac{a+\sqrt{b}}{a+\sqrt{b}}$ For this form we multiply the numerator and denominator by $a + \sqrt{b}$

$\frac{1}{a+\sqrt{b}} \times \frac{a-\sqrt{b}}{a-\sqrt{b}}$ For this form we multiply the numerator and denominator by $a - \sqrt{b}$

Example 6: Expand and simply the following expression.

$$\frac{1}{(5+\sqrt{44})} = \frac{1}{5+\sqrt{4\times 11}} = \frac{1}{5+(\sqrt{4}\times\sqrt{11})}$$
$$= \frac{1 \qquad \times (5-2\sqrt{11})}{5+2\sqrt{11} \times (5-2\sqrt{11})}$$
$$= \frac{(5-2\sqrt{11})}{25 - 10\sqrt{11} + 10\sqrt{11} - 4(11)}$$
$$= \frac{(5-2\sqrt{11})}{25-44} = \frac{-(-5+2\sqrt{11})}{-19}$$
$$= \frac{-5+2\sqrt{11}}{19}$$

Quadratics Cheat Sheet

Similar to the quadratic expression, quadratic equation can be represented in the form $ax^2 + bx + c = 0$, where a, b and c are real number and $a \neq 0$.

Solving Quadratic Equations

- To solve quadratic equations, a given equation must be rewritten in the form or kept in form of $\underline{ax^2 + bx + c} = 0$.
- Once in the correct form, the left-hand side of the equation, underlined in red, must be factorised and the factors must be equated to 0. For example, if the factorisation came to $(x + c)(x + d) = 0$, to solve for $'x'$ you have to set the factor $(x + c) = 0$ and the factor $(x + d) = 0$ and find the values of x for each respective case.

Note that quadratic equations can only have one, two or no real solutions.

Example 1: Solve the following quadratic equation.

$$3x^2 - 2x - 8 = 0$$

Factorising
$$3x^2 - 6x + 4x - 8 = 0$$
$$3x(x-2) + 4(x-2) = 0$$

Solving for values of x
$$(3x + 4)(x - 2) = 0$$
$$\therefore x = -\frac{4}{3} \quad and \quad x = 2$$

You may come across quadratic equations that may seem impossible to solve through factorisation. In this scenario, we could utilise the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 2: Solve the following equation using the quadratic formula.

$$2x^2 - 8x + 3 = 0$$
$$a = 2, b = -8 \text{ and } c = 3$$

- Substitute the values of a, b and c into the quadratic formula

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 2 \times 3}}{2 \times 2}$$

$$x = \frac{4 + \sqrt{10}}{2} \qquad x = \frac{4 - \sqrt{10}}{2}$$

Completing the square

Rewriting equations or expressions by completing the square, can be applied to many different applications in maths. Hence this would be regularly used in your further studies.

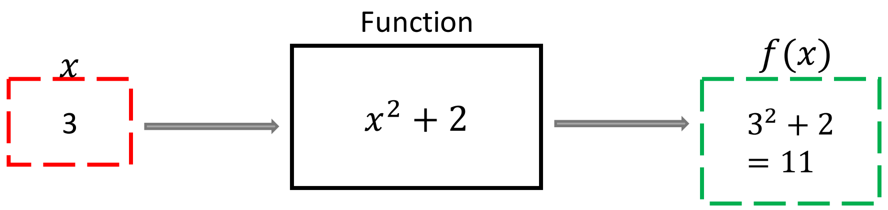
- $x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$
-

Example 3: Write the following expression in the form of $(x + a)^2 + b$

$$x^2 + 6x + 2$$
$$\left(x + \frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 2$$
$$(x + 3)^2 - 7$$

Functions

A function can be seen as a machine that takes in an input, converts this value mathematically and gives an output. The input is most commonly denoted with the term $'x'$ and the output is most commonly represented as $f(x)$ or $g(x)$. For a given function, the set of possible inputs is called domain and the set of possible outputs is called range.



During your mathematics course you will come across the questions or statements which are related to 'finding the roots of a function'. The roots of a function are the values of the input x for which the output $f(x)$ is equal to 0 ($\therefore f(x) = 0$).

Example 3: Find the value $f(2)$ of the following function and find the roots of the function.

a) $f(x) = 2x^2 + 5x - 3$
$$f(2) = 2(2)^2 + 5(2) - 3 = 15$$

b) $f(x) = 2x^2 + 5x - 3$

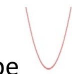
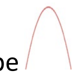
To find the root of a function we have to equate the output to 0.
$$f(x) = 0$$
$$2x^2 + 5x - 3 = 0$$
$$(2x - 1)(x + 3) = 0$$
$$2x - 1 = 0 \text{ or } x + 3 = 0$$

Hence the roots of the function are:
$$x = \frac{1}{2} \text{ and } x = -3$$

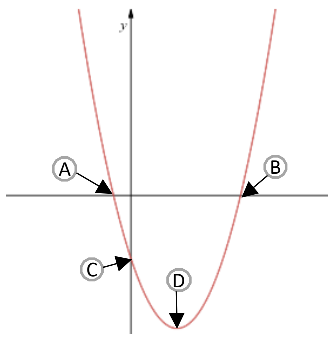
Quadratic Graphs

For functions which come in the form of a quadratic expression, the plot of $y = f(x)$ would be illustrated on a graph in the form of a shape called a parabola.

For a given quadratic function $f(x) = ax^2 + bx + c$, if

- a is positive the shape of the parabola would be 
- a is negative the shape of the parabola would be 

For a given quadratic graph, points A and B are the roots of the function as this is where the graph intercepts the $x$axis and at these two points the output $y = 0$.



At point C on the graph is where the $y = c$ as $x = 0$ at this point. In other words, $f(0) = c$.

Point D on the graph represents the 'turning' or 'stationery' point. The coordinates of the turning point of the graph can be found by completing the square

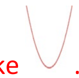
$$f(x) = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

Hence the turning point would be at $\left(-\frac{b}{2}, -\left(\frac{b}{2}\right)^2 + c\right)$

Example 4: Sketch the graph of $f(x) = 2x^2 + 5x - 3$. Label all the intercepts and turning point.

Before we start sketching, we need key pieces of information about the characteristics of the graph, which can be obtained by doing some calculations using the methods you have learnt so far.

We can compare the function to a general function of $f(x) = ax^2 + bx + c$ to determine the shape of the parabola.

$a = 2$ and $2 > 0$ hence the shape of the graph would look like .

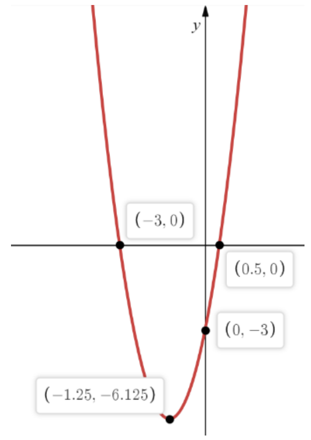
The next step would be to find the roots of the functions so we can determine where it would cross the x axis. To do this we need to solve the quadratic equation of
$$2x^2 + 5x - 3 = 0$$
$$(2x - 1)(x + 3) = 0$$
$$2x - 1 = 0 \text{ or } x + 3 = 0$$

Hence the roots of the function are:
$$x = \frac{1}{2} \text{ and } x = -3$$

At coordinates $\left(\frac{1}{2}, 0\right)$ & $(-3, 0)$ is where the x intercepts will be located.

The last bit of information we need is the location of turning point for which we need to complete the square.

$$f(x) = 2x^2 + 5x - 3$$
$$f(x) = 2\left[x^2 + \frac{5}{2}x - \frac{3}{2}\right]$$
$$f(x) = 2\left[\left(x + \frac{5}{4}\right)^2 - \frac{25}{16} - \frac{3}{2}\right]$$
$$f(x) = 2\left(x + \frac{5}{4}\right)^2 - \frac{49}{8}$$



Hence the turning point would have a coordinate of $\left(-\frac{5}{4}, -\frac{49}{8}\right)$.

Using all the information that we just obtained we can sketch the graph.

The Discriminant

The expression $b^2 - 4ac$ is called the discriminant. The value obtained using the discriminant expression on a quadratic function will indicate how many roots a given function $f(x)$ has.

For the quadratic function $f(x) = ax^2 + bx + c$

- If the discriminant $b^2 - 4ac > 0$, then the function $f(x)$ has two distinct real roots.
- If the discriminant $b^2 - 4ac = 0$, then the function $f(x)$ has one repeated real root.
- If the discriminant $b^2 - 4ac < 0$, then the function $f(x)$ has no real roots.

Example 5: Find the range of values of k for which $x^2 + 2x + k = 0$ has two distinct real roots.

$$x^2 + 2x + k = 0$$
$$a = 1, b = 2 \text{ and } c = k$$
$$b^2 - 4ac > 0 \text{ for two distinct real roots}$$
$$2^2 - 4 \times 1 \times k > 0$$
$$2^2 - 4k > 0$$
$$-4k > -4$$
$$k < 1$$

Equations and Inequalities Cheat Sheet

This chapter covers both linear and quadratic simultaneous equations and how to solve them algebraically. You should also be able to interpret solutions of a given equation graphically. It also cover both linear and quadratic inequalities.

Linear Simultaneous Equations

Linear simultaneous equations have two same unknowns in their respective equation and has one set of values between them which makes both the equations valid.

① $2x + y = 6$

② $6x + 2y = 24$

① $2 \times 6 + (-6) = 6$

② $6 \times 6 + 2 \times -6 = 24$

In the two equations 1 and 2, x and y are two unknowns. For the equations to be valid, the x value in equation 1 has to be the same x value in equation 2; the same would apply for the unknown y.
The only set of values for which this would be true is if the unknowns have the values of $x = 6$ & $y = -6$

In order to solve these simultaneous equations and find the set of values which make the given equations valid, we can use either the method of elimination or substitution.

Example 1: Solve the following simultaneous equation.

$$1)\ 2x + y = 6$$
$$2)\ 6x + 2y = 24$$

We can use the method of elimination to solve this. To eliminate one unknown, we need the coefficient of a single unknown to be the same for both the equations. In order to achieve this, we can multiply the first equation by 2. This would give use a third equation which is equivalent to equation 1:

$$3)\ 4x + 2y = 12$$

Now we can eliminate the unknown y by taking away equation 3) from 2).

② $6x + 2y = 24$
③ $4x + 2y = 12$ −

$$2x + 0 = 12$$

$$2x = 12$$
$$x = 6$$
$$1)\ \ 2 \times 6 + y = 6$$
$$y = -6$$

Quadratic Simultaneous Equations

You may come across simultaneous equations where one equation is quadratic, and one equation is linear. For this scenario you will need to use the method of substitution. As a quadratic equation is involved there can be up to two sets of values/solutions for the simultaneous equation.

Example 2: Solve the simultaneous equation.

$$1)\ y^2 + 2x = 10$$
$$2)\ 2x + y + 2 = 0$$

Equation 2 can be rewritten as:

$$2)\ x = -1 - \frac{y}{2}$$

We can substitute this rewritten equation into equation 1).

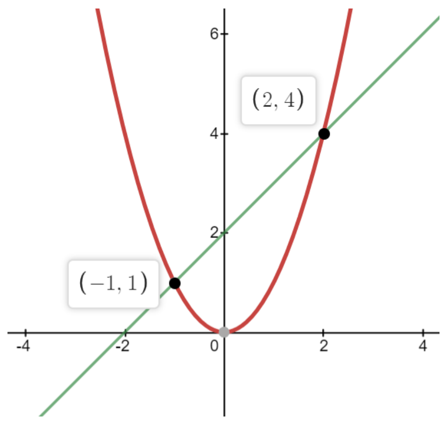
$$1)\ y^2 + 2\left(-1 - \frac{y}{2}\right) = 10$$
$$y^2 - 2 - y = 10$$
$$y^2 - y - 12 = 0$$
$$(y + 3)(y - 4) = 0$$
$$y = -3 \text{ and } y = 4$$

Substitute these two y values into equation 2) to obtain the x values.

$$x = -1 - \frac{-3}{2} = \frac{1}{2} \text{ and } x = -1 - \frac{4}{2} = -3$$

Simultaneous Equations on graphs

The solutions of a set of simultaneous equation can be represented on a graph. Simultaneous equations share the same set of values for the unknowns, hence if two given simultaneous equations were illustrated on a graph then at some point on their respective plot, they would share the same coordinate and hence intersect. Hence, the intersection point on a graph of two lines would be the solution or at least one of the solutions for the curves' or lines' simultaneous equation.



The graph on the left shows the plot of $y = x^2$ and $y - x - 2 = 0$.
If we take the point (2,4), we can check if this set of values are valid for both the equations.

$y = x^2 \quad (4) = (2)^2$ ✓
$y - x - 2 = 0 \quad (4) - (2) - 2 = 0$ ✓

e can also check if the point $(-1,1)$ agrees th the two equations.

$y = x^2 \quad (1) = (-1)^2$ ✓
$y - x - 2 = 0 \quad (1) - (-1) - 2 = 0$ ✓

Hence from this we can see that we can use graphs to solve simultaneous equations by plotting and observing any intersection points.

- Note that when solving simultaneous equations that come in the form of a quadratic equation $ax^2 + bx + c = 0$, the discriminant of the equation after substituting can be used to determine the number of solutions that the simultaneous equations have. Hence, on a graph it can also indicate the number of intersection points.

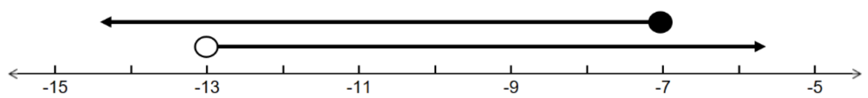
Linear Inequalities

Similar to the methods we have learnt to solve linear equations, we can also solve linear inequality problems using the same approach. When you solve an inequality, you find the set of all real numbers that make the inequality valid.
Example 3: Find the set of values for which

$$2x - 5 < 3x + 8 \quad and \quad 3x + 9 \le x - 5$$

$2x - 5 < 3x + 8$
$-13 < x$
$x > -13$

$3x + 9 \le x - 5$
$2x \le -14$
$x \le -7$



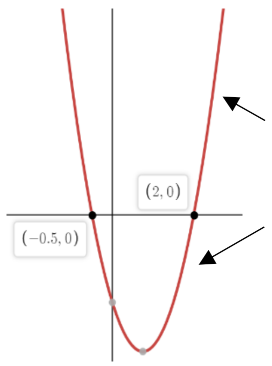
We can plot this inequality on a number line to find the set of values which agree with both the inequalities. The area which overlaps on the number line is between -13 and -7. Hence the solution for this inequality would be $-13 < x \le -7$.

Quadratic Inequalities

Similar to the method we used to solve quadratic equations, we can also solve quadratic inequalities.
Let us look at the inequality $2x^2 - 3x - 2 > 0$. To solve this, we need to first find the critical (similar to roots of a function) values of this inequality by solving the quadratic equation on the left-hand side.

$$2x^2 - 3x - 2 > 0$$
$$(2x + 1)(x - 2) > 0$$

Hence one critical point is at $x = -\frac{1}{2}$ and the other one at $x = 2$. Now we can plot the graph of $2x^2 - 3x - 2$.

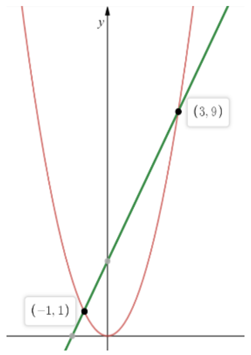


The set of values which corresponds to the inequality $2x^2 - 3x - 2 > 0$, are the x values of the plot which are above the x axis. Hence, the solution to this inequality would be $x > 2$ and $x < -\frac{1}{2}$.

However, if the inequality were to be $2x^2 - 3x - 2 < 0$, then the set of values which would correspond to it, would be the x values that are below the x axis. Hence, the solution to this inequality would be $-\frac{1}{2} < x < 2$

Inequalities on graphs and Regions

You may come across question where you are asked to find the solutions to the inequality by interpreting the functions graphically.
Example 4: The graph shows the plot of $y = x^2$ and $y = 2x + 3$. Determine the solutions to the inequality $2x + 3 > x^2$.

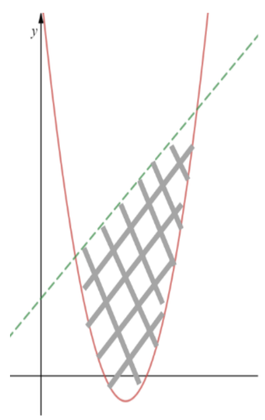


First you will need to find the points of intersection, which can be done by equating the two equations and solving it.

$$x^2 = 2x + 3$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$
$$x = 3 \text{ and } x = -1$$

Hence the two points of intersection are (3,9) and $(-1,1)$. The set of values that validates to the inequality $2x + 3 > x^2$ is the line $y = 2x + 3$ is above the curve $y = x^2$. Hence, the set of values lie between the intersection points $-1 < x < 3$

Regions on graphs can be shaded to identify the areas that satisfies given linear or quadratic inequalities.
Example 5: Shade the regions which satisfy the inequalities.



$$x^2 - 8x + 15 \le y$$
$$y - x < 3$$

The graph of $f(x) = x^2 - 8x + 15$ and $f(x) = x + 3$ is plotted.

- If $y > f(x)$, then this would represent the region above the curve or line.
- If $y < f(x)$, then this would represent the region below the curve or line.

Therefore, for the inequality $y - x < 3$, the region satisfied is represented by the area below the green dotted line and for the inequality $x^2 - 8x + 15 \le y$, the area above the red curve. Hence the region satisfied for both the inequalities is illustrated by the grey shaded area.

Graphs and Transformation Cheat Sheet

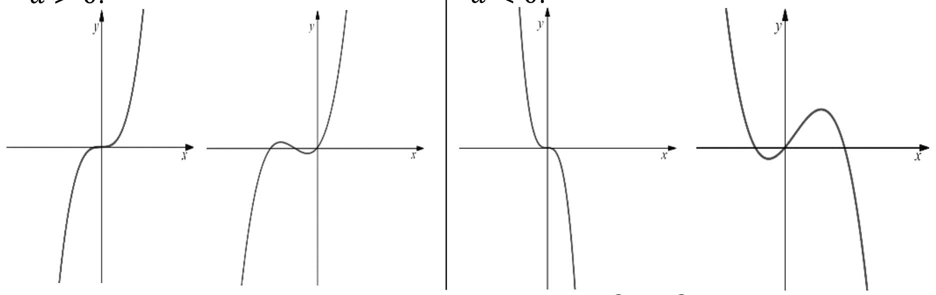
Edexcel Pure Year 1

Cubic & Quartic Graphs

Cubic functions come in the form of $f(x) = ax^3 + bx^2 + cx + d$. Quartic functions come in the form of $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d & e are all real numbers and where $a \neq 0$.

Similar to sketching quadratic graphs, cubic graphs can also be represented on a plot. There are two main basic shapes for cubic graphs if

$a > 0$: $a < 0$:

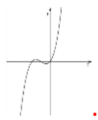


Example 1: Sketch the curve with the equation $y = x^3 + x^2 - 2x$.

Before we start sketching, we need to know

- the general shape of the cubic equation
- the location of the roots of the equation

We can compare the function to a general function of $f(x) = ax^3 + bx^2 + cx + d$ to determine the shape of the curve.

$a = 1$ and $1 > 0$ hence the shape of the graph would look like  .

The next step would be to find the roots of the functions so we can determine where it would cross the x axis. To do this we need to solve the quadratic equation of

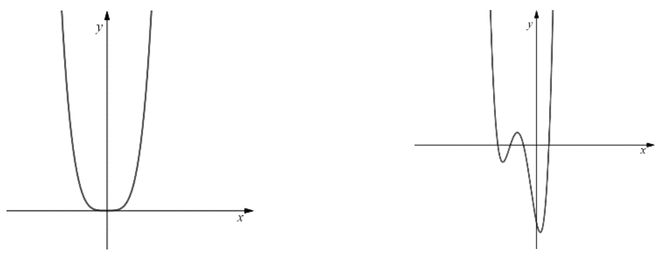
$$x^3 + x^2 - 2x = 0$$
$$x(x^2 + x - 2) = x(x + 2)(x - 1) = 0$$
$$x = 0 \ \ and \ \ x - 1 = 0 \ \ and \ \ x + 2 = 0$$

Hence the roots of the function are at:

$$x = 0 \ and \ x = 1 \ and \ x = -2$$

At coordinates $(0,0)$, $(1,0)$ & $(-2,0)$ are where the x intercepts will be located.

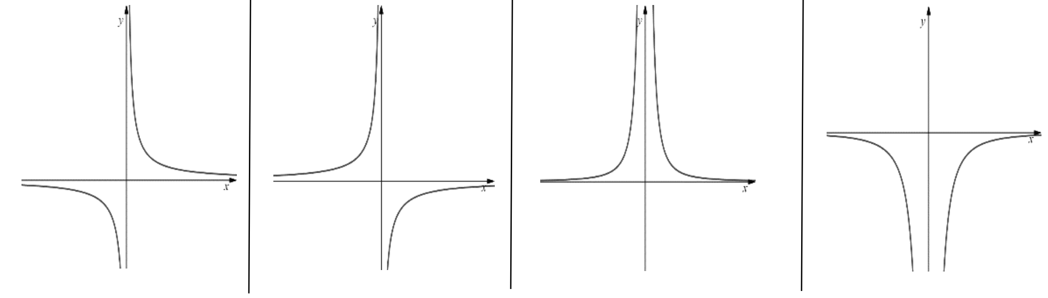
The same method can be applied to sketching graphs of quartic functions. The basic shapes of these graphs come in the form of:



Reciprocal graphs

Reciprocal graphs that come in the form of $f(x) = \frac{a}{x}$ or $f(x) = \frac{a}{x^2}$, where a is any real number, can also be sketched by considering their asymptotes. The graphs of $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$ both have asymptotes at $x = 0$ and $y = 0$. The basic shapes of these reciprocal graphs can be illustrated as:

$y = \frac{a}{x}$, where $a > 0$ | $y = \frac{a}{x}$, where $a < 0$ | $y = \frac{a}{x^2}$, where $a > 0$ | $y = \frac{a}{x^2}$, where $a < 0$



Points of Intersection

Multiple functions can be sketched on a single graph to show the points of intersection, which represent the solutions to respective equations.

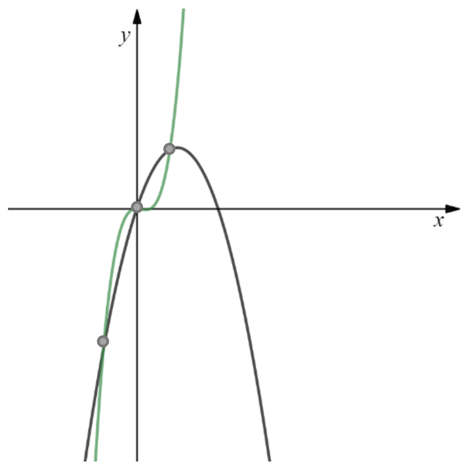
Example 2: Sketch the following functions and find the x coordinate of the intersection points.

$f(x) = 3x - x^2$ $\qquad$ $g(x) = 2x^3 - x^2$

The curves have been sketched using the methods you have learnt in this course. To find the intersection point, we must obtain the x coordinate of the locations. To solve this, we need to find the solutions to

- $f(x) = g(x)$

$$3x - x^2 = 2x^3 - x^2$$
$$2x^3 - 3x = 0$$
$$x(2x^2 - 3) = 0$$
$$x = 0 \ \ \& \ 2x^2 - 3 = 0$$
$$x = 0, x = \sqrt{\frac{3}{2}} \ \& \ x = -\sqrt{\frac{3}{2}}$$



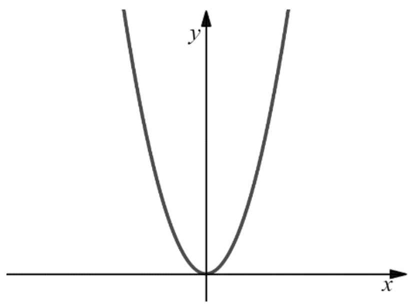
Translating graphs

Adding or subtracting a constant outside, $y = f(x) + a$, or inside, $y = f(x + a)$, a function can translate a graph vertically or horizontally respectively. Note that when translating functions, the asymptote of that function is also translated if it has one.

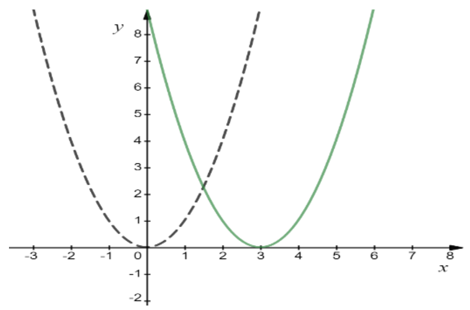
- The translation of $y = f(x) + a$ can be represented by the vector $\binom{0}{a}$
- The translation of $y = f(x + a)$ can be represented by the vector $\binom{-a}{0}$

Example 3: Given that $f(x) = x^2$, sketch the curve of $y = f(x - 3)$.

This is the graph of $y = x^2$



Applying the translation of $y = f(x - 3)$ would shift the graph by three units to the right, forming a new graph of $y = (x - 3)^2$



Stretching Graphs

Similar to translating graphs, multiplying a constant outside, $y = af(x)$, or inside, $y = f(ax)$, a function stretches the graph in the vertical direction or horizontal direction respectively.

- $y = af(x)$ would stretch the graph in the vertical direction by a multiple of a.
- $y = f(ax)$ would stretch the graph in the horizontal direction by a multiple of $\frac{1}{a}$.
- $y = -f(x)$ would be the reflection of $y = f(x)$ in the x-axis.
- $y = f(-x)$ would be the reflection of $y = f(x)$ in the y-axis.

Example 4: Given that $f(x) = 16 - 4x^2$, sketch the curve with the equation $y = \frac{1}{2}f(x)$.

Before we start to sketch $y = \frac{1}{2}f(x)$, we need to know the x and y intercepts of the curve $y = f(x)$ and how the curve looks like.

$$y = 16 - 4x^2$$
$$y = (4 - 2x)(4 + 2x)$$

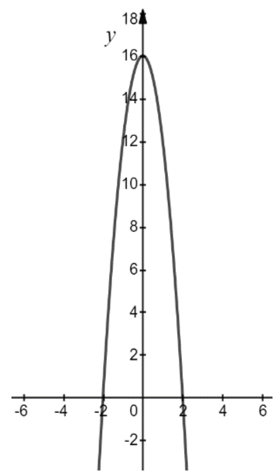
To find the x intercept we need to equate y as 0.

$$0 = (4 - 2x)(4 + 2x)$$
$$x = 2 \ and \ x = -2$$

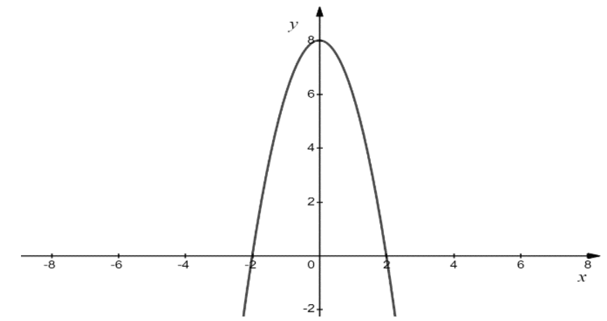
To find the y intercept we need to substitute x as $0 \therefore x = 0$.

$$y = 16 - 3 \times 0^2$$
$$y = 16$$

Hence, the curve of $y = f(x)$ is:



The transformation of $y = \frac{1}{2}f(x)$ would stretch the graph by a multiple of ½. Hence the new y intercept would be at $(16 \times \frac{1}{2}, 0) \rightarrow (8,0)$. The x intercept would not change as this transformation only effects the vertical direction.

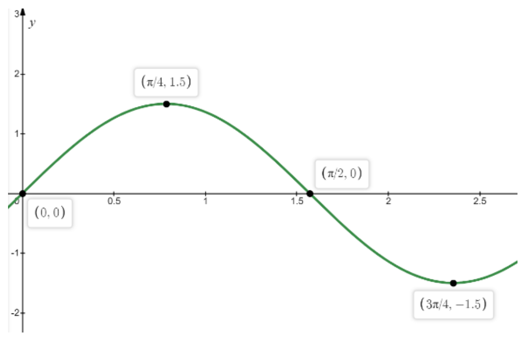


Transforming graphs

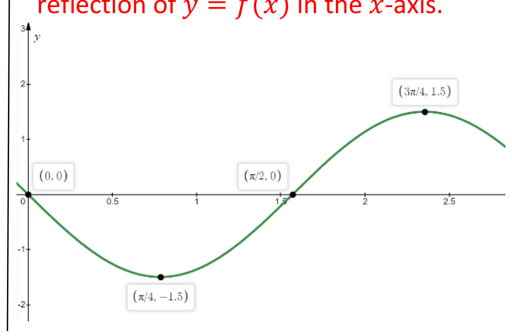
You may come across graphs with functions that are difficult to recognise. You can still apply various transformations to these types of functions by using key points such as intersection points, turning points and the x & y intercepts.

Example 5: The graph of $y = f(x)$ is given. Sketch the graph of $y = -f(x)$

The graph of $y = f(x)$



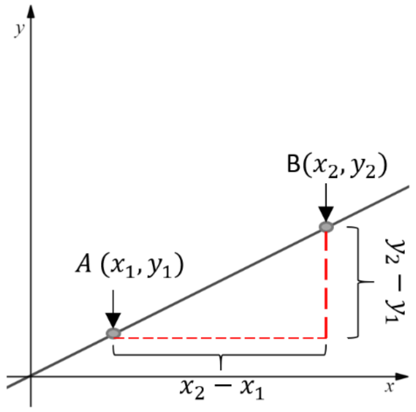
The graph of $y = -f(x)$ would be the reflection of $y = f(x)$ in the x-axis.



Straight Line Graphs Cheat Sheet

$y = mx + c$

The gradient m of straight-line graphs can be found by taking two random points A (x_1, y_1) and B (x_2, y_2) on the line and considering the horizontal and vertical distance between these two points.



The formula to calculate any given straight line's gradient is:

- $m = \frac{y_2 - y_1}{x_2 - x_1}$

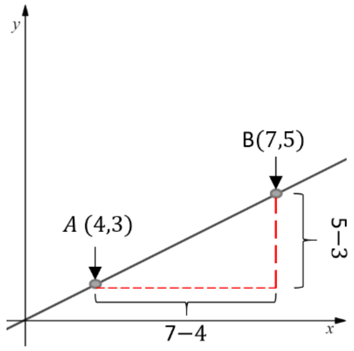
Straight line equations come in the form of

- $y = mx + c$, *where m is the gradient and c is the y intercept.*

Or

- $ax + by + c = 0$, *where a, b & c are all real numbers.*

Example 1: Calculate the gradient of the following line. Point A has coordinate (4,3) and point B has coordinate (7,5). The line passes through both point A and B.



To calculate any gradient, we need know any two points which the line passes through. We are given two points, point A (4,3) and point B (7,5). The vertical distance between point A and B needs to be divided by the horizontal distance between the points.

- $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{7-4} = \frac{2}{3}$

Example 2: A line l has gradient 2 and y intercept at $(0, -7)$. The line has equation $ax + by + c = 0$. Find the values of a, b and c.

Straight line equations come in the form of $y = mx + c$, where m is the gradient and c is the y intercept. In the question we are given that the gradient is 2, hence $m = 2$, and that the y intercept is at $(0, -7)$, hence $c = -7$. Substituting these values into $y = mx + c$ we get

$$y = 2x - 7$$

However, the question asks us to find the values of a, b and c on the form of $ax + by + c = 0$. Hence, we need to rearrange our equation into this form.

$$2x - y - 7 = 0$$

Therefore, the solutions to this question are $a = 2$, $b = -1$ and $c = -7$.

Equations of Straight Lines

Instead of relying on using $y = mx + c$ to find the equation of a line, in some cases it may be useful to find the equation of a line using the following method.

If you know one point on the line and the gradient or two distinct points on the line, we can find the equation of the line by using the formula:

- $y - y_1 = m(x - x_1)$

Example 3: Find the equation of the line that passes through the points (7,2) and (8,5).

First, we need to work out the gradient m

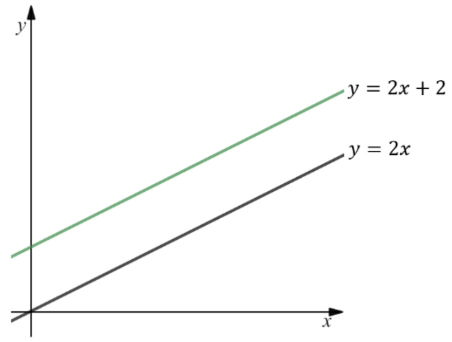
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-2}{8-7} = \frac{3}{1} = 3$$

We can take the point (7,2) or (8,5) to use in the formula. Let us take point (7,2):

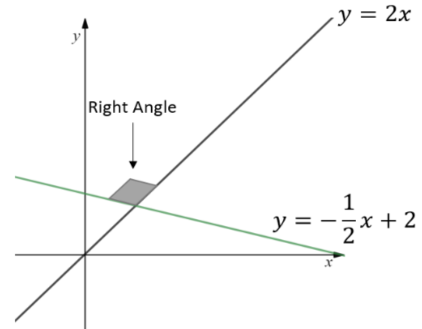
$$y - y_1 = m(x - x_1)$$
$$y - 2 = 3(x - 7)$$
$$y - 2 = 3x - 21 \rightarrow y = 3x - 19$$

Parallel and Perpendicular lines

- Parallel lines have the same gradient
- Perpendicular lines are normal to each other. In other words, they make right angles when they intersect.



The lines $y = 2x + 2$ and $y = 2x$ are parallel as they have the same gradient $m = 2$.



The lines $y = -\frac{1}{2}x + 2$ and $y = 2x$ are perpendicular. If you know the gradient m_1 of one of the lines, then the gradient of the line perpendicular to it is

- $m_2 = -\frac{1}{m_1}$

Example 4: Line l_1 is perpendicular to the line $y = -\frac{2}{3}x + 4$ and passes through the point (4,6). Find the equation of line l_1.

To find the gradient of the line l_1, we need to use the rule $m_2 = -\frac{1}{m_1}$. The gradient of the line that it is perpendicular to is $m_1 = -\frac{2}{3}$

Hence, the gradient of l_1 is

$$m_2 = -\frac{1}{m_1} = -\frac{1}{-\frac{2}{3}} = \frac{3}{2}$$

The line passes through the point (4,6)

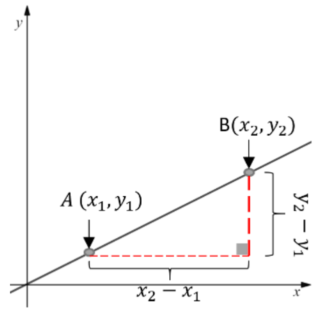
$$y - y_1 = m(x - x_1)$$
$$y - 6 = \frac{3}{2}(x - 4) \rightarrow 2y - 12 = 3x - 12$$
$$y = \frac{3}{2}x$$

Length and Area

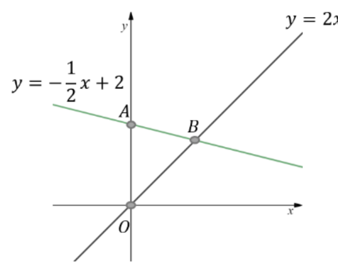
We can calculate the distance between two points on the line by using the concept of Pythagoras theorem

To calculate the distance d between points A and B we can use the formula

- $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Example 5: l_1 has equation $y = -\frac{1}{2}x + 2$ and l_2 has equation $y = 2x$. Lines l_1 & l_2 are perpendicular and intersect at point B. Find this intersection point and area of the triangle ABO.



The intersection point of the lines are at B

$$2x = -\frac{1}{2}x + 2 \rightarrow \frac{5}{2}x = 2 \rightarrow x = \frac{4}{5} \quad y = 2 \times \frac{4}{5} = \frac{8}{5} \qquad B\left(\frac{4}{5}, \frac{8}{5}\right)$$

Length AB is $AB = \sqrt{\left(\frac{4}{5} - 0\right)^2 + \left(\frac{8}{5} - 2\right)^2} = \frac{2\sqrt{5}}{5}$

Length OB is $OB = \sqrt{\left(\frac{4}{5} - 0\right)^2 + \left(\frac{8}{5} - 0\right)^2} = \frac{4\sqrt{5}}{5}$

Hence, the area of the triangle ABO is $\frac{1}{2} \times AB \times OB = \frac{1}{2} \times \frac{2\sqrt{5}}{5} \times \frac{4\sqrt{5}}{5} = \frac{4}{5}$

Algebraic Methods Cheat sheet

In this chapter you will learn about Algebraic fractions and constructing mathematical proofs

Algebraic Fractions:
Fractions whose numerator and denominator are algebraic expressions are called algebraic fractions

Simplifying algebraic fractions:
To simplify algebraic fractions you will have to cancel common factor. But sometimes, you have to factorise the expression before you cancel common factor.

Example 1:

a)

$\frac{8x^4-4x^3+6x}{2x}$ ← Divide each numerator by $2x$

$= \frac{8x^4}{2x} - \frac{4x^3}{2x} + \frac{6x}{2x}$

$= 4x^3 - 2x^2 + 3$

b) $\frac{(x+4)(3x-1)}{(3x-1)}$ ← Cancel the common factor of $(3x-1)$

$= x+4$

With Factorise

C)

Factorise — Cancel Common Factor

$\frac{x^2+3x+2}{x^2+5x+4} = \frac{(x+2)(x+1)}{(x+4)(x+1)} = \frac{x+2}{x+4}$

Dividing polynomials
A polynomial is a finite expression with positive whole number indices (≥ 0)

Polynomials	Not polynomials
$3x+5, 3x^2y+5y+6, 8$	$\sqrt{x}$, $5x^{-2}$, $\frac{4}{x}$

You can use long division to divide polynomial by $(x \pm p)$, where p is a constant

Example 2: Write the polynomial $4x^3+9x^2-3x-10$ in the form $(x \pm p)(ax^2+bx+c)$ by dividing

$$\begin{array}{r} 4x^2 \\ x+2\overline{)4x^3+9x^2-3x-10} \\ 4x^3+8x^2 \\ \hline x^2-3x \end{array}$$

Start by dividing the first term by x, so that $4x^3 \div x = 4x^2$

Multiply $(x+2)$ by $4x^2$
So that $4x^2 \times (x+2) = 4x^3 + 8x^2$

Subtract,
So that $(4x^3+9x^2) - (4x^3+8x^2) = x^2$
And copy $-3x$

$$\begin{array}{r} 4x^2+x-5 \\ x+2\overline{)4x^3+9x^2-3x-10} \\ 4x^3+8x^2 \\ \hline x^2-3x \\ x^2+2x \\ \hline -5x-10 \\ -5x-10 \\ \hline 0 \end{array}$$

Repeat the process till you get a remainder

If the remainder is 0 then the divisor, in this case $(x+2)$ is a factor of polynomial $4x^3+9x^2-3x-10$

Hence, $4x^3+9x^2-3x-10 = (x+2)(4x^2+x-5)$

The factor theorem:
The factor theorem is a quick way of finding simple linear factor of a polynomial

The factor theorem states that if $f(x)$ is a polynomial then,

- If $f(p) = 0$, then $(x-p)$ is a factor of $f(x)$
- If $(x-p)$ is a factor of $f(x)$, then $f(p) = 0$

Example 3: $f(x) = 3x^3 - 12x^2 + 6x - 24$

a) Use factor theorem to show that $(x-4)$ is a factor of $f(x)$
b) Hence, show that 4 is the only real root of the equation $f(x) = 0$

a)
According to the theorem,

If $(x-4)$ is a factor of $3x^3 - 12x^2 + 6x - 24$, then $f(4)$ must be equal to 0

Substitute $x = 4$ in the polynomial
$f(x) = 3x^3 - 12x^2 + 6x - 24$
$\therefore f(4) = 3(4)^3 - 12(4)^2 + 6(4) - 24$
$= 192 - 192 + 24 - 24$
$= 0$
So $(x-4)$ is a factor of $3x^3 - 12x^2 + 6x - 24$

b)
To find the root of the equation, first you need to use long division to factorise the polynomial and equate it to 0

$$\begin{array}{r} 3x^2+6 \\ x-4\overline{)3x^3-12x^2+6x-24} \\ 3x^3-12x^2 \\ \hline 6x-24 \\ 6x-24 \\ \hline 0 \end{array}$$

$f(x) = (x-4)(3x^2+6)$

Equate $f(x) = 0$

$(x-4)(3x^2+6) = 0$

$3x^2+6$ is a quadratic equation
$\Rightarrow a = 3, b = 0, c = 6$
And to check if the roots are real or not, you need to find discriminant i.e. $b^2 - 4ac$
If $b^2 - 4ac < 0 \Rightarrow$ equation has no real roots
By substituting the values of a, b and c in the discriminant we get,
$b^2 - 4ac = 0 - 4(3)(6) = -72 < 0$
Hence $3x^2 + 6$ has no real roots. Therefore $f(x)$ has only one real root of $x = 4$

Mathematical proof:
Key terms:

Theorem	Mathematical statement (or a Conjecture)
A statement that has been proven	A statement that has yet to be proven

In this section you will have to prove mathematical statement (or conjecture). In simple words you will have to show that the mathematical statement is true in specified cases. You will have to use the following steps to prove a statement

Start with know facts or theorems ⇨ Clearly show logical steps ⇨ Statement of proof

Example 4: Prove that $n^2 - n$ is an even number for all values of n.

You know the fact that ODD×EVEN = EVEN.

Start by writing $n^2 - n$ as multiple of two terms. We can do that by factorising the term as follows
$n^2 - n = n(n-1)$

Any number is either ODD or EVEN. Now consider if n is even, then $n-1$ must be odd which implies
$n \times (n-1) \Rightarrow$ Even × Odd = Even
If $n-1$ is even then n must be odd which implies
$n \times (n-1) \Rightarrow$ Odd × Even = Even

Hence, $n^2 - n$ is even for all values of n

Prove an Identity:
Identical statements mean they are always equal mathematically. In this section you will have to prove an identity. That is, you will have to show the right hand side of the equation equal to left hand side.

Example 5: Prove that $(x+\sqrt{y})(x-\sqrt{y}) \equiv x^2 - y$

Start by solving one side of the identity. It will be logical to start with $(x+\sqrt{y})(x-\sqrt{y})$ as this can be expanded.

$(x+\sqrt{y})(x-\sqrt{y}) = x(x-\sqrt{y}) + \sqrt{y}(x-\sqrt{y})$
$= x^2 - x\sqrt{y} + x\sqrt{y} - (\sqrt{y}\sqrt{y})$ as $\sqrt{y} \times \sqrt{y} = y$
$= x^2 - y$

$\Rightarrow (x+\sqrt{y})(x-\sqrt{y}) \equiv x^2 - y$
Hence, we have proved the identity.

Methods of proof:
There are different methods to prove a mathematical statement. However, in this chapter you will only learn Proof by Exhaustion.

Proof by Exhaustion: In this method you will have to split your statement into smaller cases and prove each case separately. This way you will be able to prove that the statement is true.

Example 6: Prove that the sum of two consecutive square numbers between 1^2 and 8^2 is an odd number.

You will prove this by exhaustion. Start by listing all square numbers between 1^2 and 8^2 and add the consecutive square numbers to get a result,

$2^2 + 3^2 = Odd, 3^2 + 4^2 = Odd, 4^2 + 5^2 = odd, 5^2 + 6^2 = odd, 6^2 + 7^2 = Odd$
Now you can see, each case is proved to be an odd number
So, the sum of two consecutive square numbers between 1^2 and 8^2 is always an odd number.

Counter-example:
You can prove a mathematical statement is not true by counter-example. A counter-example is one example that does not work for the given statement. To disprove a statement one counter example is enough.

Example 7: Show, by means of a counter-example, that the following inequality does not hold when p and q are both negative

$$p + q > \sqrt{4pq}$$

Start by taking negative values for both p and q
$p = -1, q = -2$
$p + q = (-1) - (-2) = -1 + 2 = 1$
$\sqrt{4pq} = \sqrt{4(-1)(-2)} = \sqrt{8}$
But $1 < \sqrt{8}$, i.e. $p + q < \sqrt{4pq}$
Hence by counter example, we proved the inequality is not true for negative values

The Binomial Expansion Cheat Sheet

The binomial expansion can be used to expand brackets raised to large powers. It can be used to simplify probability models with a large number of trials, such as those used by manufacturers to predict faults.

Pascal's triangle

You can use Pascal's triangle to quickly expand expressions such as $(x+2y)^3$. Consider the expansions of $(a+b)^n$ for n = 0,1,2,3 and 4:

$(a+b)^0 = 1$

$(a+b)^1 = 1a + 1b$

$(a+b)^2 = 1a^2 + 2ab + 1b^2$

$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

Each coefficient is the sum of the 2 coefficients immediately above it

Every term in the expansion of $(a+b)^n$ has total index n:
In the $6a^2b^2$ term the total index is $2+2=4$.
In the $4ab^3$ term the total index is $1+3=4$.

Pascal's triangle is formed by adding adjacent pairs of the numbers to find the numbers on the next row.

Here are the first 7 rows of Pascal's triangle:

1
1 + 1
1 + 2 + 1
1 + 3 + 3 + 1
1 + 4 + 6 + 4 + 1
1 + 5 + 10 + 10 + 5 + 1
1 + 6 + 15 + 20 + 15 + 6 + 1

The third row of Pascal's triangle gives the coefficients in the expansion of $(a+b)^2$

The $(n+1)th$ row of Pascal's triangle gives he coefficients in the expansion of $(a+b)^n$.

Example 1:
Use Pascal's triangle to find the expansions of:
a. $(x+2y)^3$ b. $(2x-5)^4$

a. $(x+2y)^3$

Index = 3 so look at the 4th row of Pascal's triangle to find the coefficients.

The coefficients are 1, 3, 3, 1 so:

$(x+2y)^3 = 1x^3 + 3x^2(2y) + 3x(2y)^2 + 1(2y)^3$
$= x^3 + 6x^2y + 12xy^2 + 8y^3$

This is the expansion of $(a+b)^3$ with $a=x$ and $b=2y$. Use brackets to ensure you don't make a mistake.

b. $(2x-5)^4$

Index = 4 so look at the 5th row of Pascal's triangle.

The coefficients are 1, 4, 6, 4, 1 so:

$(2x-5)^4 = 1(2x)^4 + 4(2x)^3(-5) + 6(2x)^2(-5)^2 + 4(2x)(-5)^3 + 1(-5)^4$
$= 16x^4 - 160x^3 + 600x^2 - 1000x + 625$

This is the expansion of $(a+b)^4$ with $a=2x$ and $b=-5$

Example 2:
The coefficient of x^2 in the expansion of of $(2-cx)^3$ is 294. Find the possible values of the constant c. (Note: if there is an unknown in the expression, form an equation involving the unknown)

The coefficients are 1, 3, 3, 1:
The term in x^2 is $3\times 2(-cx)^2 = 6c^2x^2$
So, $6c^2 = 294$
$c^2 = 49 \Rightarrow c \pm 7$

Factorial notation

Combinations and factorial notation can help you expand binomial expressions. For larger indices, it is quicker than using Pascal's triangle.

Using factorial notation $3\times 2\times 1 = 3!$
You can use factorial notation and your calculator to find entries in Pascal's triangle quickly. The number of ways of choosing r items from a group of n items is written as ${}^nC_r$ or $\binom{n}{r}$:
${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

The rth entry in the nth row of Pascal's triangle is given by ${}^{n-1}C_{r-1} = \binom{n-1}{r-1}$

Example 3: Calculate
a. 5! b. ${}^5C_2$ c. the 6th entry in the 10th row of Pascal's triangle

a. $5! = 5\times 4\times 3\times 2\times 1 = 120$
b. ${}^5C_2 = \frac{5!}{2!3!} = \frac{120}{12} = 10$
c. ${}^9C_5 = 126$

The rth entry in the nth row is ${}^{n-1}C_{r-1}$

Use the ${}^nC_r$ and ! functions on your calculator to answer this question.

You can calculate ${}^5C_2$ by using the ${}^nC_r$ function on your calculator.
${}^nC_r = \frac{n!}{r!\,(n-r)!} = \frac{5!}{2!\,(5-2)!}$

The binomial expansion

The binomial expansion is a rule that allows you to expand brackets. You can use $\binom{n}{r}$ to work out the coefficients in the binomial expansion. For example,
in the expansion of $(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$, to find the b^3 term you can choose multiples of b from 3 different brackets. You can do this in $\binom{5}{3}$ ways so the b^3 term is $\binom{5}{3}a^2b^3$.

The binomial expansion is:
$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n$

where $\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$

Example 4 : Use the binomial theorem to find the expansion of $(3-2x)^5$.

$(3+2x)^5 = 3^5 + \binom{5}{1}3^4(-2x) + \binom{5}{2}3^3(-2x)^2 + \binom{5}{3}3^2(-2x)^3 + \binom{5}{4}3^1(-2x)^4 + (-2x)^5$

$= 243 - 810x + 1080x^2 - 720x^3 + 240x^4 - 32x^5$

There will be 6 terms. Each term has a total index of 5. Use $(a+b)^n$ with $a=3$, $b=-2x$ and $n=5$

Solving Binomial Problems

You can use the general term of the binomial expansion to find individual coefficients in a binomial expansion.

In the expansion of $(a+b)^n$ the general term is given by $\binom{n}{r}a^{n-r}b^r$.

Example 6:
a. Find the coefficient of x^4 in the binomial expansion of $(2+3x)^{10}$.

x^4 term = $\binom{10}{4}2^6(3x)^4$
$= 210\times 64\times 81x^4$
$= 1088640x^4$

The coefficient of x^4 in the binomial expansion of $(2+3x)^{10}$ is 1088640.

b. Find the coefficient of x^3 in the binomial expansion of $(2+x)(3-2x)^7$.

$(3-2x)^7$

First, find the first four terms of the binomial expansion of $(3-2x)^7$

$= 3^7 + \binom{7}{1}3^6(-2x) + \binom{7}{2}3^5(-2x)^2 + \binom{7}{3}3^4(-2x)^3 + \cdots$
$= 2187 - 10206x + 20412x^2 - 22680x^3 + \cdots$
$\Rightarrow (2+x)(2187 - 10206x + 20412x^2 - 22680x^3 + \cdots)$

Now expand the brackets $(2+x)(3-2x)^7$

x^3 term $= 2\times(-22680x^3) + x \times 20412x^2$
$= -24948x^3$

The coefficient of x^3 in the binomial expansion of $(2+x)(3-2x)^7$is -24948.

There are 2 ways of making the x^3 term: (constant term $\times\, x^3$ term) and (x term $\times\, x^2$ term)

Binomial Estimation

If the value of x is less than 1, then x^n gets smaller as n gets larger. If x is small you can sometimes ignore large powers of x to approximate a function or estimate a value.

Example 9:
a. Find the first four terms of the binomial expansion, in ascending powers of x, of $\left(1-\frac{x}{4}\right)^{10}$.

$\left(1-\frac{x}{4}\right)^{10} = 1^{10} + \binom{10}{1}1^9\left(-\frac{x}{4}\right) + \binom{10}{2}1^8\left(-\frac{x}{4}\right)^2 + \binom{10}{3}1^7\left(-\frac{x}{4}\right)^3 + \cdots$

$= 1 - 2.5x + 2.8125x^2 - 1.875x^3 + \cdots$

b. Use your expansion to estimate the value of 0.975^{10}, giving your answer to 4 decimal places.

We want $\left(1-\frac{x}{4}\right) = 0.975$
$\frac{x}{4} = 0.025$
$x = 0.1$

Calculate value of x

Substitute $x = 0.1$ into the expansion for $\left(1-\frac{x}{4}\right)^{10}$ from part a:
$0.975^{10} \approx 1 - 0.25 + 0.028125 - 0.001875$
$= 0.77625$
$0.975^{10} \approx 0.7763$ to 4 d.p

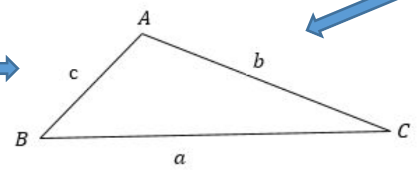
Using a calculator, $0.975^{10} = 0.77632962$. so, approximation is correct.

Trigonometric ratios Cheat Sheet

Edexcel Pure Year 1

The cosine rule:
The cosine rule can be used to find missing side and missing angle. The rule can be rearranged in two ways depending on what we need to find, missing side or missing angle

Where a, b, and c are lengths opposite to angles A, B and C respectively.



Find missing side:

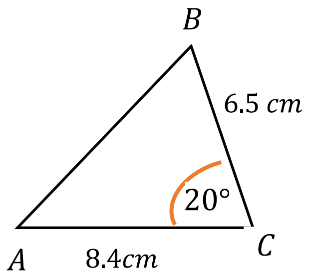
$$a^2 = b^2 + c^2 - 2bc\cos A$$

This version of the rule is used to find a missing side if you know two sides and the angle between them.

Finding missing angle.

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

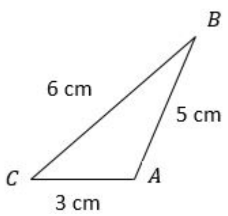
This version of the rule is used to find missing angle given all three sides.
Example 1: calculate the length of the missing side



The missing length is AB which is opposite to angle C.
Use the cosine rule for missing side substitute values of a, b and c
Let $a = 6.5cm$, $b = 8.4\ cm$ and $AB = c = ?$

$c^2 = a^2 + b^2 - 2ab\cos C$
$AB^2 = 6.5^2 + 8.4^2 - 2 \times 6.5 \times 8.4 \times \cos 20° = 10.1955$
$AB = \sqrt{10.1955 \ldots} = 3.19cm$

Example 2: Find the size of the smallest angle in a triangle whose side have length 3cm, 5cm and 6cm



Start by drawing the triangle and label it say ABC. The smallest angle is opposite is to the smallest side so angle B is the required angle. Use the cosine rule for missing angle and substitute values of a, b and c.

$a = 6cm, b = 3cm,$ and $c = 5cm$
$b^2 = a^2 + c^2 - 2ac\cos B$

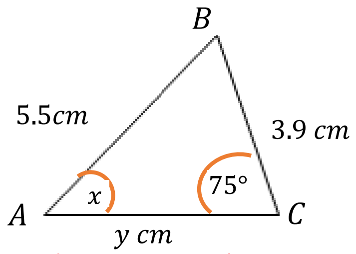
$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{6^2+5^2-3^2}{2\times 6 \times 5} = 0.866 \ldots \ldots$$

$B = \cos^{-1} 0.8666..$
$B = 29.9°$
Hence, the smallest angle is 29.9°

The sine Rule.
The sine rule can be used to work out missing side or angles in triangles. Similar to cosine rule, sine rule can also be rearranged in two ways to find either missing angle or missing side.
Please refer to the figure shown by arrow for the sine rule.
Where a, b, and c are lengths opposite to angles A, B and C respectively.

Finding missing side:	Finding missing angle
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Example 3: Work out the values of x and y



In this problem, there is a missing side as well as a missing angle. You will have to use both versions of sine rule.

Finding missing angle:
The side opposite to angle x is length $BC = a = 3.9\ cm$
$a = 3.9\ cm, c = 5.5\ cm, C = 75°, x = ?$
Use the sine rule for missing angle and substitute values of a, c and angle C

$$\frac{\sin A}{a} = \frac{\sin C}{c} \Rightarrow \frac{\sin x}{3.9} = \frac{\sin 75°}{5.5} \Rightarrow \sin x = \frac{3.9 \times \sin 75°}{5.5} = 0.68493$$

$x = \sin^{-1}(0.68493) = 43.23°$ ← Using sine inverse to find x

Finding missing angle:
In order to calculate, we need the angle opposite to length y which is $\angle ABC$
$\angle ABC = 180° - (75 + 43.2)° = 61.8°$
Use the sine rule for missing angle and substitute values of c, angle B and C.

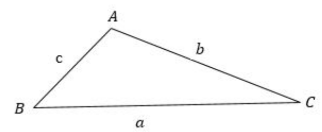
$$\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{y}{\sin 61.8°} = \frac{5.5}{\sin 75} \Rightarrow y = \frac{5.5 \times \sin 61.8°}{\sin 75} = 5.018$$

$y = 5.02\ cm$

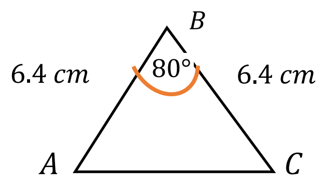
Two solutions for sine:
The sine rule sometimes produces two possible solutions for a missing angle as $\sin\theta = \sin(180° - \theta)$

Areas of triangles:
In this topic you will learn to calculate area of any triangle given 2 sides and the angle between them

$$A = \frac{1}{2}ab\ \sin C$$



Example 4: Calculate the area of triangle.
The angle between two sides AB and BC is angle B
AB is opposite to angle C so AB = c and AC is opposite to angle B so AC = b



Area$= \frac{1}{2}ac\ \sin B$
$A = \frac{1}{2} \times 6.4 \times 6.4\ \times \sin 80°$
$A = 20.16..$
$A = 20.2\ cm^2$

Solving Traingle problems:
Problems involving triangles can be solved by using sine rule, cosine rule along with pythagoras theorem and standard right-angled triangle trigonometry.

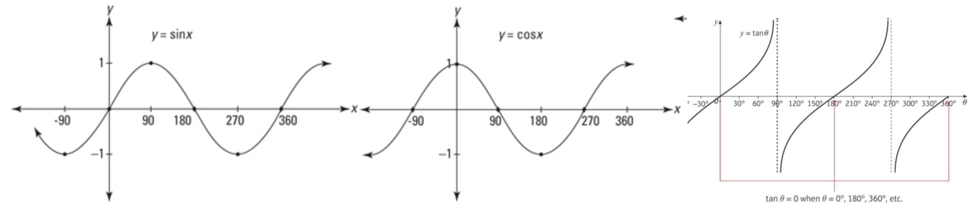
In this section you will learn when to use the above mentioned rules.
Right-angled triangle: Try using basic trigonometry and Pythagoras's theorem to work out other information
Not Right-angled triangle: Use the Sine rule or the Cosine Rule. You can use the rules depending on what information is given.

Use Sine rule	Use Cosine rule
when you are considering 2 angles and 2 sides	when you are considering 3 sides and 1 angle

Graphs of sine, cosine and tangent:
In this section you will have to sketch the graphs of sine, cosine and tangent.
All three graphs are periodic i.e. they repeat themselves after a certain interval.
The below table will help you with properties of the three graphs

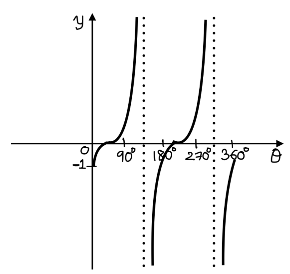
$y = \sin\theta$	$y = \cos\theta$	$y = \tan\theta$
Crosses the x axis at $..,-180°, 0, 180°, 360°,..$	Crosses the x axis at $..,-90°, 90°, 270°, 450°,..$	Crosses the x axis at $..,-180°, 0, 180°, 360°,..$
Maximum value = 1 Minimum value = -1	Maximum value = 1 Minimum value = -1	No maximum value or minimum value
		Has vertical asymptotes At $x = -90°, 90°, 270°,..$

You can refer to the graphs below for sine, cosine and tangent graphs



Transforming trigonometric graphs:
In chapter 4, you have learned transformations i.e. translation and reflection. In this section you will have to apply the knowledge of transformations in trigonometric functions and sketch the new curve.

Example 5
Sketch the graph of $y = \tan(\theta - 45°)$
The graph of $y = \tan(\theta - 45°)$ is the graph of $\tan\theta$ translated by 45° to the right.
Remember $f(x+\theta) \Rightarrow \theta$ **shifted to LEFT and** $f(x-\theta) \Rightarrow \theta$ **shifted to the RIGHT**



The graphs will shift by 45° to the right
So if $\tan\theta$ meets the θ- axis at$(0°, 0°)$ then $\tan(\theta - 45°)$ meets the θ - axis at $(0° + 45°, 0°) = (45°, 0°)$
Hence,
The graph meets the θ axis at $(45°, 0), (225°, 0)$
And to find, where the graph meets the y-axis do the following
You know that $\theta = 0°$ on y-axis,
So $y = \tan(\theta - 45°) = \tan(0 - 45°) = \tan(-45°) = -1$
Hence the graph meets the y- axis at $(0, -1)$ and has asymptotes at $\theta = 135°$and $\theta = 315°$

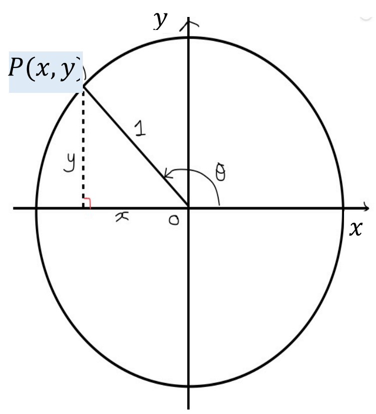
Trigonometric identities Cheat Sheet

Edexcel Pure Year 1

Angles in all four quadrants

Unit circles:

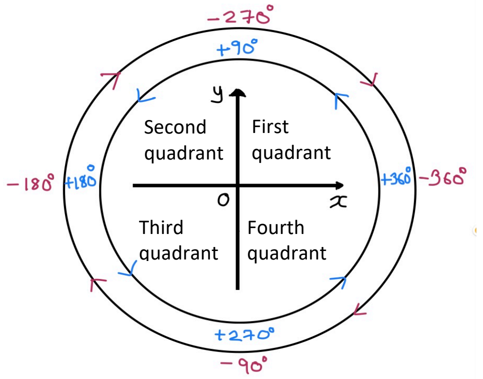
A unit circle is a circle with radius of 1 unit. It will help you understand the trigonometric ratios.



For a point $P(x, y)$ on a unit circle such that OP making an angle with the positive x-axis
$\cos\theta = x$-coordinate of P
$\sin\theta = y$- coordinate of P
$\tan\theta = \frac{y}{x}$ =gradient of OP
You always start measuring θ from positive x-axis
Positive angles ⟺ Anti-clock wise
Negative angles ⟺ Clockwise

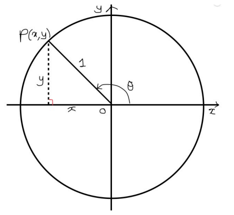
With the help of unit circle you can find values and signs of sine, cosine and tangent.

The x-y plane is divided into quadrants:



Angles may lie outside the range 0-360°, but they always lie in one of the four quadrants.
For e.g. 520° is equivalent to $520° - 360° = 160°$ which lies in second quadrant

Example 1:
Find the signs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ in the second quadrant.
Draw a circle with centre 0 and radius 1, with $P(x, y)$ in the second quadrant.

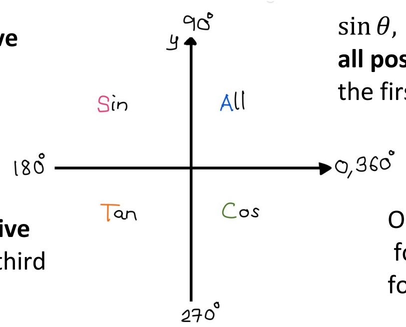


You know that x is $-ve$ and y is $+ve$ in the second quadrant
$\sin\theta = +ve$, $\cos\theta = -ve$
$\tan\theta = \frac{+ve}{-ve} = -ve$
So, only $\sin\theta$ is $+ve$ in the second quadrant

With the help of the following diagram, you can determine the signs of each of the trigonometric ratios

Only **sin θ is positive** for angle θ in the second quadrant.

$\sin\theta$, $\cos\theta$ and $\tan\theta$ are **all positive** for angle θ in the first quadrant.



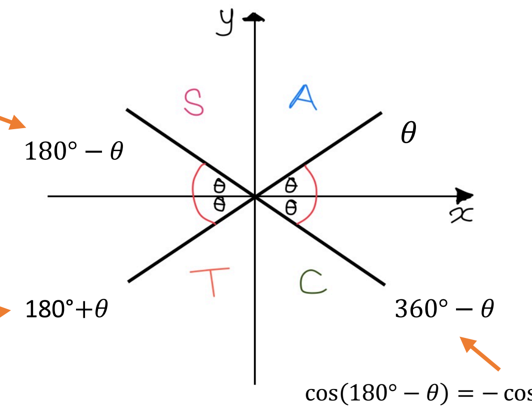
Only **tan θ is positive** for angle θ in the third quadrant

Only **cos θ is positive** for angle θ in the fourth quadrant.

You can use the following rules to find sin, cos or tan of any positive or negative angle using the corresponding acute angle made with the x-axis

$\sin(180° - \theta) = +\sin\theta$
$\sin(180° + \theta) = -\sin\theta$
$\sin(360° - \theta) = -\sin\theta$

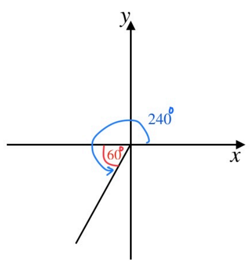
$\tan(180° - \theta) = -\tan\theta$
$\tan(180° + \theta) = +\tan\theta$
$\tan(360° - \theta) = -\tan\theta$



$\cos(180° - \theta) = -\cos\theta$
$\cos(180° + \theta) = -\cos\theta$
$\cos(360° - \theta) = \cos\theta$

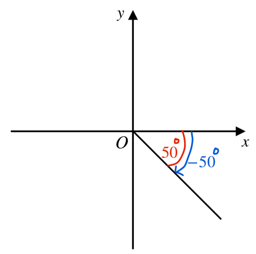
Example2:
Express the following in terms of trigonometric ratios of acute angles.
a. $\sin 240°$ b. $\cos(-50°)$

a.



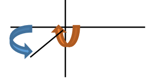
The angle 240° is obtuse and measured from the $+ve$ x-axis anti-clockwise.
So the acute angle is 60°
sin is – ve in the third quadrant
So $\sin 240° = -\sin 60°$

b.



The angle $-50°$ is the angle measured from the positive x- axis clockwise.
50° is the acute angle.
cos is $+ve$ in the fourth quadrant
So $\cos(-50°) = \cos 50°$

Example 3:
Given that θ is an acute angle, express $\tan(\theta - 540°)$in terms of $\tan\theta$
To express $\tan(\theta - 540°)$ in terms of $\tan\theta$, we need to find in which quadrant the angle $\theta - 540°$ lies.
You know that 540° is equivalent to $540° - 360° = 180°$
$\Rightarrow -540°$ is equivalent to $-180° \Rightarrow$ 180° clockwise and θ = anti-clockwise
So first you will go 180° clockwise and then θ anti-clockwise which will be in the third quadrant.
tan is +ve in the third quadrant
Hence, $\tan(\theta - 540°) = \tan\theta$



Exact values of trigonometric ratios.
You can find exact values of sin, cos and tan of 30°, 45° and 60°. Please refer the table below for the exact values.

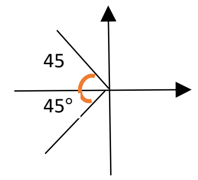
	30°	45°	60°
sin θ	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos θ	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan θ	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Trignometric Identities:
Equation of unit circle is $x^2 + y^2 = 1$
As we know $\cos\theta = x$ and $\sin\theta = y$ $\Rightarrow \cos^2\theta + \sin^2\theta = 1$
For all values of θ,$\sin^2\theta + \cos^2\theta \equiv 1$
For all values of θ, such that $\cos\theta \neq 0$, $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$
You can use the above identities to simplify trigonometric expressions and complete proofs

Example 4: Simplify $a.\ 5\sin^2 3\theta + 5\cos^2 3\theta$ $b.\ \frac{\sqrt{(1-\cos^2 x)}}{\cos x}$
a. Start by factorising the equation
$\Rightarrow 5(\sin^2 3\theta + \cos^2 3\theta\ )$
$\Rightarrow 5 \times 1 = 5$ As $\sin^2\theta + \cos^2\theta \equiv 1 \Rightarrow \sin^2 3\theta + \cos^2 3\theta = 1$
b.
$\frac{\sqrt{(1-\cos^2 x)}}{\cos x} = \frac{\sqrt{\sin^2\theta}}{\cos\theta}$ ← As $\cos^2\theta + \sin^2\theta = 1 \Rightarrow (\sin^2\theta = 1 - \cos^2\theta)$
$\Rightarrow \frac{\sqrt{(1-\cos^2 x)}}{\cos x} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

Simple Trignometric equations.
In this section you will learn to solve simple trignometric equations of the form $\sin\theta = k$, $\cos\theta = k$ (where $-1 \le k \le 1$) and $\tan\theta = p$ (where $p \in \mathbb{R}$)
$-1 \le k \le 1$ as sin and cos has maximum = 1 and minimum = −1
$p \in \mathbb{R}$ as tan has no maximum or minimum value

Example 5: Solve the equation $2\cos\theta = -\sqrt{2}$ for θ, in the interval $0 \le x \le 360°$
First rearrange the equation in the form $\cos\theta = k$
So $\cos\theta = \frac{-\sqrt{2}}{2} = -0.7071$
$\theta = \cos^{-1}(-0.7071) = 45°$
The values you get on calculator taking inverse of trigonometric functions are called principal values. But principal values will not always be a solution to the equation.



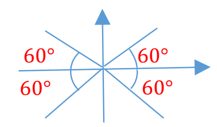
As $\cos\theta = -0.7071\ and\ \theta = 45° \Rightarrow$ cos is negative so you need to look θ in the 2nd and 4th quadrant
45° is the acute angle (i.e angle made with the horizontal axis) but we are looking for the angle made from the positive x- axis anti-clockwise.
So, there are two solutions
$180° - 45° = 135°$ and $180° + 45° = 225°$
Hence, θ= 135° or θ= 225°

Harder trigonometric equations:
You will have to solve equations of the form
$\sin n\theta = k, \cos n\theta = k\ and\ \tan n\theta = p$
$\sin(\theta + \alpha) = k, \cos(\theta + \alpha) = k\ and\ \tan(\theta + \alpha) = p$
It is same as solving simple equations, but will have some extra steps

Example 6: Solve the equation $\sin(x + 60°) = 0.3$ in the interval $0 \le x \le 360°$
Let $X = x + 60° \Rightarrow \sin X = 0.3$
The interval for X will be $0 + 60° \le X \le 360° + 60° \Rightarrow 60° \le X \le 420°$
$X = \sin^{-1} 0.3 = 17.45°$, principal value
Sin is positive which mean 17.45° should be in the 1st and 2nd quadrant.
One of the solution will be $180° - 17.45° = 162.54°$
Now the other solution could be 17.45° but $60° \le X \le 420°$, so it cannot be 17.45°.
So start from $+ve$ x-axis and measure one full circle i.e. 360° and add 17.5°
$\Rightarrow 360° + 17.45° = 377.45°$ So $X = 162.54 \ldots°, 377.45 \ldots°$
Subtract 60° from each value: Hence, $x = 102.5°\ or\ 317.5°$

Equations and Identities:
You will have to solve quadratics equations in $\sin\theta, \cos\theta\ and\ \tan\theta$

Example 7: Solve for θ, in the interval $0 \le x \le 360°$, the equation $2\cos^2\theta - \cos\theta - 1 = 0$
Start by factorising the equation as you do for quadratic equation
$2\cos^2\theta - \cos\theta - 1 = 0$ Compare with $2x^2 - x - 1 = (2x + 1)(x - 1)$
So $(2\cos\theta + 1)(\cos\theta - 1) = 0$
$\cos\theta = -\frac{1}{2}\ or\ \cos\theta = 1$ Set each factor equal to 0 thereby finding two sets of solutions
$\cos\theta = -\frac{1}{2} \Rightarrow \theta = 60°$
Cosine is negative implies solution is in the 2nd and 3rd quadrants
In the 2nd quadrant $\theta = 180 - 60 = 120°$. So, one solution is 120°
In the 3rd quadrant $\theta = 180 + 60 = 240°$.
So, the other solution in the 3rd quadrant will be 240°
$\cos\theta = 1\ so\ \theta = 0\ or\ 360°$
So the solutions are
$\theta = 0°, 120°, 240°, 360°$



Year 12
A-Level Mathematics (Applied)

Mathematics Knowledge Organisers

Autumn Term 2024

Data Collection Cheat Sheet

Population and sample

In statistics, population is the whole set of items that are of interest. Information obtained from a population is known as raw data. A census measures or observes every member of a population. A sample is a selection of observations taken from a subset of population and used to find out more information about the population as a whole.

	Advantages	Disadvantages
Census	• Results should be completely accurate	• Time consuming and expensive • Cannot be used when testing destroys process • Hard to process large quantity of data
Sample	• Less time consuming and cheaper • Fewer people have to respond • Less data needs to be processed	• Data may not be as accurate • Sample may not be large enough to give information about small subgroups of the population

Individual units of a population are known as sampling units. Sampling units are named and numbered to form a list called a sampling frame.

Random sampling

Each member of the population has an equal chance of being selected. The sample should be representative of the population and bias should be removed. There are 3 types of random sampling.

- Simple random sampling

A simple random sample of size n is one where every sample of size n has an equal chance of being selected.

Example 1: The 100 members of a yacht club are listed alphabetically in the club's membership book. The committee wants to select a sample of 12 members to fill in a questionnaire. Explain how a simple random sample can be taken using:

A) Calculator or random number generator:
Number each member from 1-100. Use a calculator or random number generator to generate 12 random numbers between 1-100. Select the members who correspond to the numbers.

B) Lottery sampling:
Write the name of members on identical cards and place them in the hat. Draw up 12 cards and select these members.

Advantages	Disadvantages
• Free of bias • Easy and cheap for small samples and populations • Each sampling unit has a known and equal chance of selection	• Not suitable for large samples and populations • Sampling frame needed

- Systematic sampling

The required elements are chosen at regular intervals from an ordered list.

Example 2: A sample of size 20 is required from a population of 100.

$100 \div 20 = 5$ so every fifth person is chosen.
The first person is chosen at random.
If the first person chosen is 2, the remaining samples will be 7, 12, 17 etc.

Advantages	Disadvantages
• Simple and quick to use • Suitable for large samples and large populations	• A sampling frame is needed • Bias introduced if sampling frame is not random

- Stratified sampling

The population is divided into mutually exclusive strata and a random sample is taken from each.

$$\text{Number sampled in a stratum} = \frac{\text{number in stratum}}{\text{number in population}} \times \text{overall sample size}$$

Example 3: A factory manager wants to find out about what his workers think about the factory canteen facilities. He decides to give a questionnaire to a sample of 80 workers. It is thought that different age groups will have different opinions.

There are 75 workers between ages 18 and 32, 140 workers between ages 33 and 47, and 85 workers between ages 48 and 62.

Explain how he can use stratified sampling to select the sample.

1. Total number of workers: $75 + 140 + 85 = 300$
2. Finding the number of workers needed from each age group:
   18-32: $\frac{75}{300} \times 80 = 20$ workers
   33-47: $\frac{140}{300} \times 80 = 37\frac{1}{3} \approx 37$ workers
   48-62: $\frac{85}{300} \times 80 = 22\frac{2}{3} \approx 23$ workers
   If the number of workers required is not a whole number, it is rounded off to the nearest whole number.
3. Number the workers in each group.
4. Use a random number generator or table to produce the required quantity of random numbers.

Advantages	Disadvantages
• Sample accurately reflects population structure • Proportional representation of group within population	• Population must be clearly classified into distinct strata • Same disadvantages as simple random sampling within each stratum

Non-random sampling

There are two types of non-random sampling that you need to know:

- Quota sampling

An interviewer or researcher selects a sample that reflects the characteristics of the whole population.

Advantages	Disadvantages
• Allows a small sample to still be representative of the population • No sampling frame required • Quick, easy and inexpensive • Easy comparison between different groups within a population	• Non-random sampling can introduce bias • Population must be divided into groups, which can be costly or inaccurate • Increasing scope of study increases number of groups, which adds time and expenses • Non-responses not recorded

- Opportunity sampling or convenience sampling

Sample is taken from people who are available at the time of study and who fits the criteria you are looking for.

Advantages	Disadvantages
• Easy and inexpensive	• Unlikely to provide a representative result • Highly dependent on individual researcher

Types of data

Variables or data associated with numerical observations are called quantitative variables or quantitative data.

Variables associated with non-numerical observations are qualitative variables or qualitative data.

A variable that can take any value in a given range is a continuous variable. A variable that can only take specific values is a discrete variable.

In a grouped frequency table, the specific data values are not shown.

- Class boundaries show the maximum and minimum values in each group or class
- The midpoint is the average of class boundaries
- The class width is the difference between upper and lower class boundaries

Large data set

If you need to do calculations on large data sets in your exam, the relevant extract will be provided.

Measures of Location and Spread Cheat Sheet

Measures of central tendency

A measure of central tendency describes the centre of the data. You need to decide of the best measure to use in particular situations.

The mode or modal class is the value of class which occurs most often. This is used when data is qualitative or quantitative with one mode or two modes (bimodal). It is not informative if each value only occurs once.

The median is the middle value when the data values are put in order. This is used for quantitative data and usually used when there are extreme values as they are unaffected.

The mean can be calculated using:

$$\bar{x} = \frac{\Sigma x}{n}$$

Where $\bar{x}$ (x bar) is the mean,
Σx is the sum of the data values,
n is the number of data values

For data given in a cumulative frequency table, the mean can be calculated using:

$$\bar{x} = \frac{\Sigma xf}{\Sigma f}$$

Where Σfx is the sum of the products of the data values and their frequencies,
Σf is the sum of frequencies

The mean is used for quantitative data. It uses all values in the data therefore it gives a true measure of data. However, it is affected by extreme values.

You can calculate the mean, class containing median and modal class for continuous data presented in a grouped frequency table by finding the midpoint of each class interval.

Other measures of location

The median(Q_2) splits the data into two equal halves (50%).
The lower quartile (Q_1) is one quarter of the way through the dataset.
The upper quartile (Q_3) is three quarters of the way through the dataset.

Percentiles split the data set into 100 parts. The 10th percentile is one-tenth of the way through the data, for example. 10% of data values are less than the 10th percentile and 90% are greater.

To find lower and upper quartiles for discrete data:

1. Divide n by 4. (lower quartile) OR
   Find $\frac{3}{4}$ of n. (upper quartile)
2. If this is a whole number, the lower or upper quartile is the midpoint between this data point and the number above. If it is not, round up and pick this number.

When data is presented in a grouped frequency table, you can use interpolation to estimate the medians, quartiles, and percentiles. This method assumes that the data values are evenly distributed within each class.

$Q_1 = \frac{n}{4}$th data value
$Q_2 = \frac{n}{2}$th data value
$Q_3 = \frac{3n}{4}$th data value

Measures of spread

Measures of spread shows how spread out the data is. They are also known as measures of dispersion or measures of variation.

- Range
  The difference between largest and smallest values in the dataset.
- Interquartile range (IQR)
  The difference between upper and lower quartile.
- Interpercentile range
  Difference between the values of two given percentiles.

Variance (σ^2) and standard deviation (σ)

The variance also shows how spread out the data is. There are 3 versions of the formulae used to find variance:

1. $\sigma^2 = \frac{\Sigma(x-\bar{x})^2}{n}$
2. $\sigma^2 = \frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2$

Easier to use when given raw data

You can remember this as "mean of squares minus square of means"

3. $\sigma^2 = \frac{S_{xx}}{n}$

Where $S_{xx} = \Sigma(x-\bar{x})^2 = \Sigma x^2 - \frac{(\Sigma x)^2}{n}$
You can use calculator to calculate S_{xx}

S_{xx} is a summary statistic used to simplify formula

Standard deviation is the square root of variance.

$$\sigma = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n}} = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2} = \sqrt{\frac{S_{xx}}{n}}$$

For grouped data presented in frequency table:

$$\sigma^2 = \frac{\Sigma f(x-\bar{x})^2}{\Sigma f} = \frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2$$

$$\sigma = \sqrt{\frac{\Sigma f(x-\bar{x})^2}{\Sigma f}} = \sqrt{\frac{\Sigma fx^2}{\Sigma f} - \left(\frac{\Sigma fx}{\Sigma f}\right)^2}$$

Where f is the frequency of each group and Σf is the total frequency

Example 1: Shamsa records the time spent out of school during lunch hour to the nearest minute, x, of the students in her year in the table below. Calculate the standard deviation.

Time spent out of school (min)	35	36	37	38
Frequency	3	17	29	34

1. Find Σfx^2, Σfx and Σf
   $\Sigma fx^2 = 3 \times 35^2 + 17 \times 36^2 + 29 \times 37^2 + 34 \times 38^2$
   $= 114504$
   $\Sigma fx = 3 \times 35 + 17 \times 36 + 29 \times 37 + 34 \times 38$
   $= 3082$
   $\Sigma f = 3 + 17 + 29 + 34 = 83$
2. Use formula for grouped data in frequency table to find variance:
   $\sigma^2 = \frac{114504}{83} - \left(\frac{3082}{83}\right)^2 = 0.74147\ldots$
3. Square root variance to find standard deviation:
   $\sigma = \sqrt{0.74147\ldots} = 0.861$ (3s.f.)

Coding

Each value in the data can be coded to give a new set of values, which is easier to work with. Coding also changes different statistics in different ways.

If data is coded using the formula $y = \frac{x-a}{b}$, where a and b are constants that you have to choose or given in the question:

- Mean of coded data: $\bar{y} = \frac{\bar{x}-a}{b}$
  Rearrange the formula to find original mean: $\bar{x} = b\bar{y} + a$
- Standard deviation of coded data: $\sigma_y = \frac{\sigma_x}{b}$
  Rearrange the formula to find original standard deviation: $\sigma_x = b\sigma_y$

Example 2: A scientist measures the temperature, x°C, at five different points of a nuclear reactor. Her results are given below:
332°C, 355°C, 306°C, 317°C, 340°C

a. Use the coding $y = \frac{x-300}{10}$ to code this data.

Substitute each value into x to get coded data, y.

Original data, x	332	355	306	317	340
Coded data, y	3.2	5.5	0.6	1.7	4.0

b. Calculate the mean and standard deviation of the coded data.
   $\Sigma y = 15$, $\Sigma y^2 = 59.74$
   $\bar{y} = \frac{15}{5} = 3$
   $\sigma_y^2 = \frac{59.74}{5} - \left(\frac{15}{5}\right)^2 = 2.948$
   $\sigma_y = \sqrt{2.948} = 1.72$ (3s.f.)

c. Calculate the mean and variance of the original data using your answers from part b.
   $3 = \frac{\bar{x}-300}{10}$ so $\bar{x} = 330$°C
   $1.72 = \frac{\sigma_x}{10}$ so $\sigma_x = 17.2$°C (3s.f.)

Modelling in Mechanics

Constructing a model

Mechanics deals with motion and action of forces on objects. Mathematical models can be constructed to simulate real-life situations, but in many cases it is necessary to simplify the problem by making assumptions so that it can be described using equations or graphs in order to solve it.

Example 1: The motion of a basketball as it leaves a player's hand and passes through the net can be modelled using the equation $h = 2 + 1.1x - 0.1x^2$, where h m is the height of the basketball above the ground and x m is the horizontal distance travelled.

a. Find the height of the basketball :

i. When it is released

$x = 0\,; h = 2 + 0 + 0$
$Height = 2\text{m}$

ii. At a horizontal distance of 0.5m

$x = 0.5\,; h = 2 + 1.1 \times 0.5 - 0.1 \times (0.5)^2$
$Height = 2.525 \text{ m}$

b. Use the model to predict the height of the basketball when it is at a horizontal distance of 15m from the player.

$x = 15\,; h = 2 + 1.1 \times 15 - 0.1 \times (15)^2$
$Height = -4 \text{ m}$

c. Comment on the validity of this prediction.

Height cannot be negative so the model is not valid when $x = 15$ m.

Modelling assumptions

Modelling assumptions can simplify a problem and allow you to analyse the real-life situation using known mathematical techniques. These assumptions will affect the calculations in a particular problem.

Some common models and modelling assumptions

Model	Modelling assumptions
Smooth surface	Assume there is no friction between the surface and any object on it
Rough surface	Objects in contact with the surface experience a frictional force if they are moving or are acted on by a force
Air resistance – Resistance experienced as an object moves through the air	Usually modelled as being negligible
Gravity – Force of attraction between all objects. Acceleration dur to gravity is denoted by g, where the value of g= 9.8 ms^{-2}	• Assume that all objects with mass are attracted towards the Earth • Earth's gravity is uniform and acts vertically downwards • g is constant and is taken as 9.8 ms^{-2}, unless otherwise stated in the question

Quantities and units

The International System of Units, (abbreviated as SI) is the modern form of the metric system.

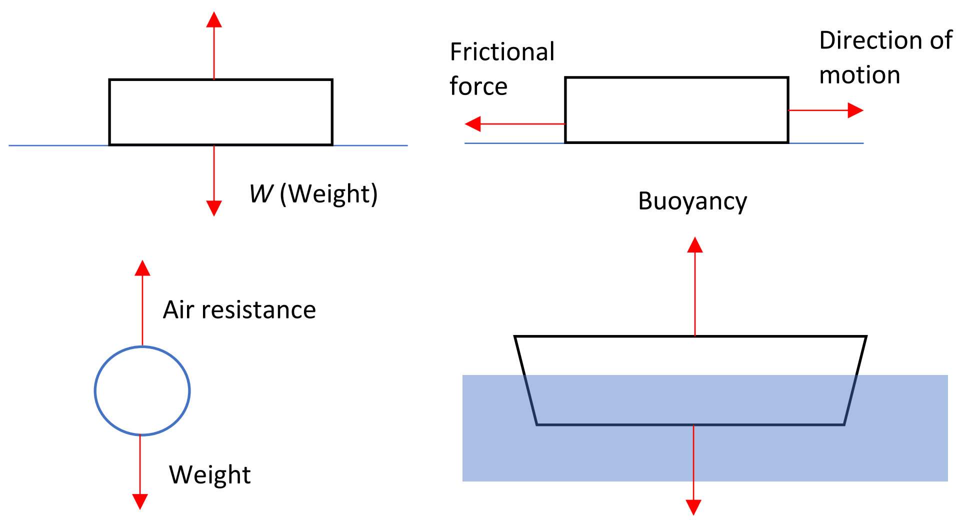
These **base** SI units are most commonly used in mechanics.

Quantity	Unit	Symbol
Mass	Kilogram	kg
Length/displacement	Metre	m
Time	Seconds	s

These **derived** units are compound units built from the base units.

Quantity	Unit	Symbol
Speed/velocity	Metres per second	ms^{-1}
Acceleration	Metres per second per second	ms^{-2}
Weight/force	Newton	N (= kg ms^{-2})

Some of the common force diagrams that you will encounter in mechanics :



Meanings of each of the above forces:

- The **weight** (or gravitational force) of an object acts vertically downwards
- The **normal reaction** is the force acting perpendicular to a surface when an object is in contact with the surface.
- The **friction** is a force which opposes the motion between two rough surfaces
- **Buoyancy** is the upward force on a body that allows it to float or rise when submerged in a liquid.
- **Air resistance** opposes motion of an object falling towards the ground.

Example 2: Write the following quantities in SI units.

a. 4km
$4 \text{ km} = 4 \times 1000 = 4000 \text{ m}$

b. 0.32 grams
$0.32 \text{ g} = 0.32 \div 1000 = 3.2 \times 10^{-4} \text{ kg}$

c. $5.1 \times 10^6 \text{ km h}^{-1}$
$5.1 \times 10^6 \text{ km h}^{-1} = 5.1 \times 10^6 \times 1000$
$= 5.1 \times 10^9 \text{ m h}^{-1}$
$5.1 \times 10^9 \div (60 \times 60) = 1.42 \times 10^6 \text{ m s}^{-1}$

Working with vectors

Vector quantities are quantities which have both magnitude and direction. Vector quantities can be positive or negative. Examples are:

Quantity	Description	Unit
Displacement	Distance in a particular direction	Metre (m)
Velocity	Rate of change of displacement	Metres per second (ms^{-1})
Acceleration	Rate of change of velocity	Metres per second per second (ms^{-2})

Scalar quantities are quantities which have magnitude only. Scalar quantities are always positive. Examples are:

Quantity	Description	Unit
Distance	Measure of length	Metre (m)
Speed	Measure of how quickly a body moves	Metres per second (ms^{-1})
Time	Measure of ongoing events taking place	Second (s)
Mass	Measure of the quantity of matter contained in an object	Kilogram (kg)

You can also describe vectors using **i-j** notation, where **i** and **j** are the unit vectors in the positive x and y directions.

Example 3: The velocity of a particle is given by v = 3**i** + 5**j** ms^{-1}. Find:

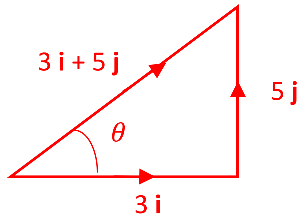
a. The speed of the particle

$|\text{speed}| = |v| = \sqrt{3^2 + 5^2} = \sqrt{34}$
$= 5.83 \text{ ms}^{-1}$

b. The angle the direction of motion of the particle makes with the unit vector **i**

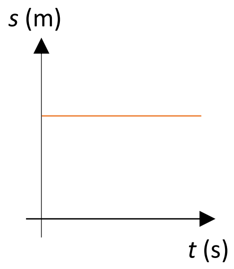
Angle made with i = θ

$\tan\theta = \frac{5}{3} \;\; so\; \theta = 59°$

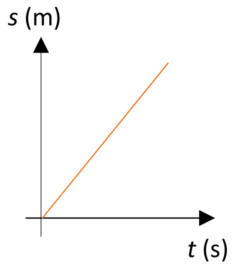


Displacement-time graphs

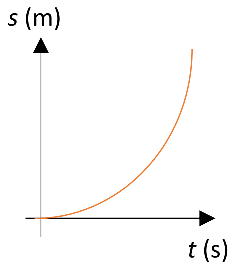
- Displacement is always plotted on the vertical axis and time on the horizontal axis.
- In these graphs *s* represents the displacement of an object from a given point in metres and *t* represents the time taken in seconds.



- *No change in displacement over time*
- *Object is stationary*



- *Displacement increases at a constant rate over time*
- *Object is moving with constant velocity*



- *Displacement is increasing at greater rate as time increases*
- *Velocity is increasing and object is accelerating*

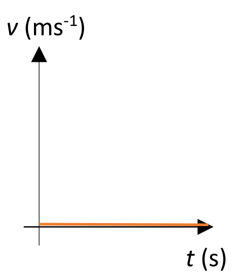
Velocity is the rate of change of displacement. Gradients of displacement-time graphs represent velocity.

Average velocity = $\frac{\text{displacement from starting point}}{\text{time taken}}$

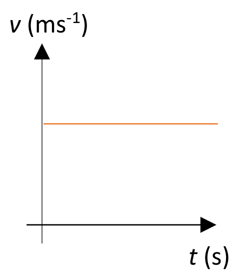
Average speed = $\frac{\text{total distance travelled}}{\text{time taken}}$

Velocity-time graphs

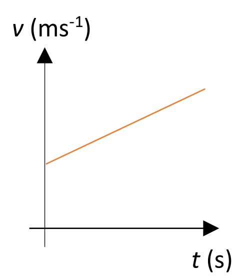
- Velocity is always plotted on the vertical axis and time on the horizontal axis.
- In these graphs *v* represents the velocity of an object in metres per second and *t* represents the time taken in seconds.



Object is stationary



Object moves with constant velocity

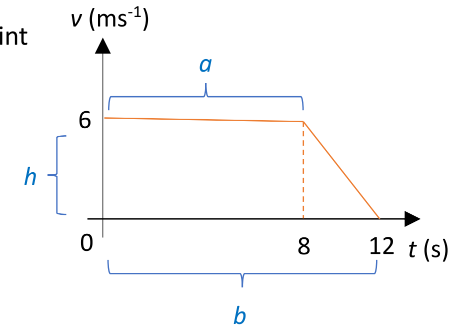


Object moves with increasing velocity at a constant rate (ie. constant acceleration)

Acceleration is the rate of change of velocity, represented by gradients of velocity-time graphs. The area under the graph of velocity time graph represents distance travelled.

Example 1 : The figure shows a velocity-time graph illustrating the motion of a cyclist for a period of 12 seconds. She moves at a constant speed of 6 ms^{-1} for the first 8 secs. She then decelerates at a constant rate, stopping after a further 4 secs.

a. Find the displacement from the starting point of the cyclist after this 12 secs period.



Displacement = area under the graph

$s = \frac{1}{2}(a+b)h$

$= \frac{1}{2}(8+12)6$

$= 10 \times 6 = 60$ m

b. Work out the rate at which the cyclist decelerates.

Acceleration is the gradient of the slope. Find the deceleration between 8*s* to 12*s*.

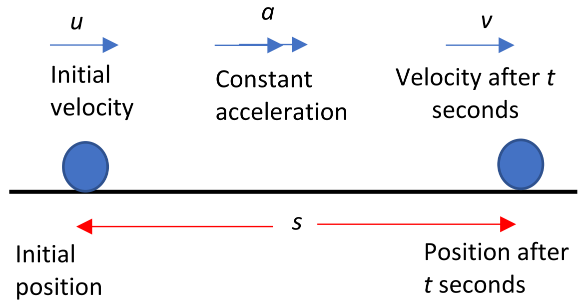
$a = \frac{0-6}{12-8}$

$= \frac{-6}{4} = -1.5$ ms^{-2}

Constant acceleration formulae 1

A standard set of letters is used for the motion of an object moving in a straight line with constant acceleration.

- *s* is the displacement
- *u* is the initial velocity
- *v* is the final velocity
- *a* is the acceleration
- *t* is the time

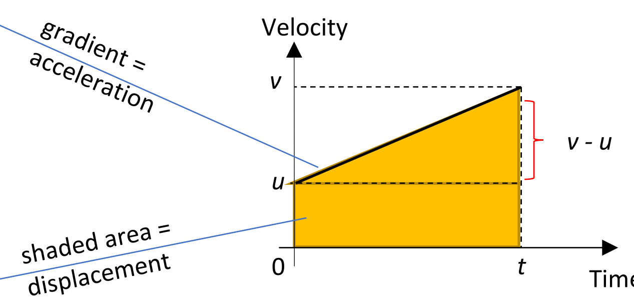


- $a = \frac{v-u}{t}$

Rearrangement of the equation above gives us :

- $\boldsymbol{v = u + at}$

$\boldsymbol{s = (\frac{u+v}{2})t}$



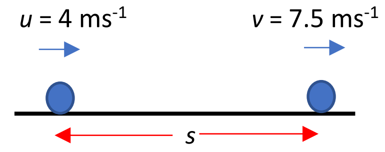
The formulae in the red box are often used to solve any questions. Choosing the appropriate formulae depends on which information is given by the question.

Example 2: A cyclist is travelling along a straight road. She accelerates at a constant rate from a velocity of 4 ms^{-1} to velocity of 7.5 ms^{-1} in 40 seconds. Find:

a. The distance she travels in these 40 seconds

$s = (\frac{u+v}{2})t$

$= \left(\frac{4+7.5}{2}\right) \times 40 \quad = 230$ m



b. Her acceleration in these 40 seconds

$v = u + at$

$7.5 = 4 + a\,(40)$

$a = \frac{7.5-4}{40} = 0.0875$ ms^{-2}

Constant acceleration formulae 2

You can derive another 3 formulae from the previous formulae $v = u + at$ and $s = (\frac{u+v}{2})t$. This will give you another 3 formulae which are:

- $v^2 = u^2 + 2as$
- $s = ut + \frac{1}{2}at^2$
- $s = vt - \frac{1}{2}at^2$

You need to know how these formulae are derived

Example 3: A particle is moving from A to B with constant acceleration 5 ms^{-2}. The velocity of the particle at A is 3 ms^{-1} in the direction of A to B. The velocity of the particle at B is 18 ms^{-1} in the same direction. Find the distance from A to B.

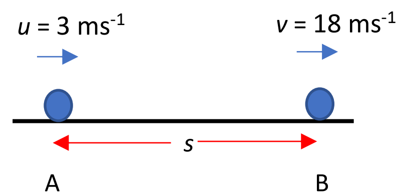
$v^2 = u^2 + 2as$

$18^2 = 3^2 + 2\,(5) \times s$

$324 = 9 + 10s$

$s = \frac{324-9}{10}$

$s = 31.5$ m



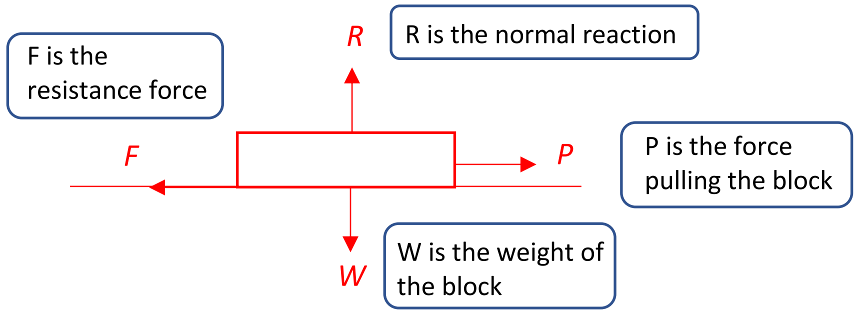
Vertical motion under gravity

When an object is free falling (moves down vertically under gravity) towards the earth, the acceleration is constant, independent of the weight/mass of the object. Ignoring the air resistance, any object which falls under gravity or in vacuum will have an acceleration due to gravity which is often represented as g= 9.8 ms^{-2}. A downward vertical motion has a positive g value while an upward motion caused by gravity (eg. an object bouncing upward) will have g= - 9.8 ms^{-2}. The negative value indicates that the object is moving an opposite direction (upwards) from the gravity.

Force Diagrams

A force diagram is a diagram showing all the forces acting on an object. Each force is shown as an arrow pointing in the direction in which the force acts. Force diagrams are used to model problems involving forces.

Example 1 : A block of weight W is being pulled to the right by a force, P, across a rough horizontal plane. Draw a force diagram to show all the forces acting on the block.



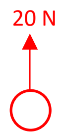
When the forces acting upon an object are balanced, the object is said to be in equilibrium. You can find the resultant force by adding forces acting in the same direction and subtracting forces in opposite directions.

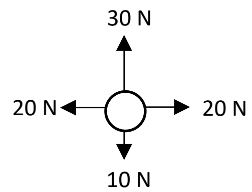
Newton's first law of motion states that an object at rest will stay at rest and that an object moving with constant velocity will continue to move with constant velocity unless an unbalanced force acts on the object.

A resultant force will cause the object to accelerate in the same direction as the resultant force.

Example 2: The diagram shows the forces acting on a particle.

a. Draw the resultant force





b. Describe the motion of the particle.
The particle is accelerating upwards

Forces as vectors

Forces can be written as vectors using **i-j** notation or as column vectors. Resultant of 2 or more forces can be given as vectors by adding the vectors. An object in equilibrium has a resultant vector force of 0**i** + 0**j**.

Example 3: The forces 2**i** + 3**j**, 4**i** – **j**, -3**i** + 2**j** and *a***i** + *b***j** act on an object which is in equilibrium. Find the values of a and *b*.

$(2\mathrm{i}+3\mathrm{j})+(4\mathrm{i}-\mathrm{j})+(-3\mathrm{i}+2\mathrm{j})+(a\,\mathrm{i}+b\,\mathrm{j})=0$

$(2+4-3+a)\mathrm{i}+(3-1+2+b)\mathrm{j}=0$

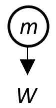
$\Rightarrow\quad 3+a=0 \quad \text{and} \quad 4+b=0$

$\Rightarrow\quad a=-3 \quad \text{and} \quad b=-4$

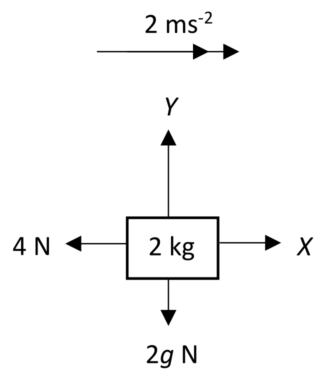
Forces and acceleration

Newton's second law of motion states that the force needed to accelerate a particle is equal to the product of the mass of the particle and the acceleration produced: *F* = *ma*

Gravity is the force between any object and the Earth. The force due to gravity acting on an object is called the weight of the object, acting vertically downwards. A body falling freely experiences an acceleration of g = 9.8 ms^{-2}. Hence, free fall objects have equations of *W* = mg.



Example 4: In the diagram below, the body is accelerating as shown. Find the magnitudes of the unknown forces *X* and Y.



a. Horizontal forces:

$X-4=2\times 2$

$X=8\text{ N}$

b. Vertical forces:

$Y-2g=2\times 0$

$Y=2\,(9.8)$

$Y=19.6\text{ N}$

Motion in 2 dimensions

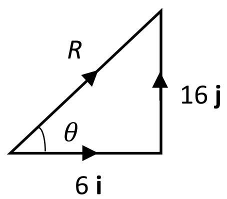
You can use *F* = *ma* to solve problems involving vector forces acting on particles.

Example 5: In this question **i** represents the unit vector due east, and **j** represents the unit vector due north. A resultant force of (3**i** + 8**j**) N acts upon a particle of mass 0.5 kg.

a. Find the acceleration of the particle in the form (*p***i** + *q***j**) ms^{-2}.

$F=ma$
$(3i+8j)=0.5\times a$
$a=2(3i+8j)$
$a=(6i+16j)\mathrm{ms}^{-2}$

b. Find the magnitude of R and bearing of the acceleration of the particle.



$|R|=\sqrt{6^2+16^2}=2\sqrt{73}\text{ N}=17.1\text{ N (1 d.p.)}$

$\tan\theta=\dfrac{16}{6}\quad so\ \theta=69.4°\ \text{(1 d.p.)}$

So the bearing of the acceleration is

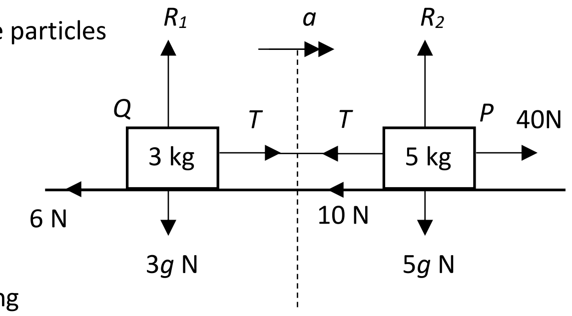
$90°-69.4°=020.6°$

Remember bearings are always measured clockwise from north

Connected particles

If a system involves the motion of more than one particle, the particles may be considered separately. However, if all parts of the system are moving in the **same straight line**, then you can also treat the whole system as a single particle.

Example 6: Two particles, *P* and *Q*, of masses 5kg and 3kg respectively, are connected by a light inextensible string. Particle *P* is pulled by a horizontal force of magnitude 40N along a rough horizontal plane. Particle *P* experiences a frictional force of 10N and particle Q experiences a frictional force of 6N.



a. Find the acceleration of the particles
For the whole system :
All horizontal forces :
$40-10-6=8a$
$8a=24$
$a=3\text{ ms}^{-2}$

b. Find the tension of the string
For P (horizontal forces):
$40-T-10=5\times 3$
$T=15\text{N}$

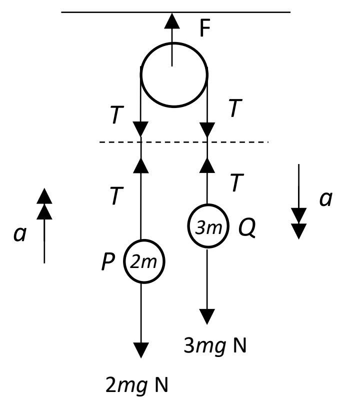
Newton's third law states that for every action there is an equal and opposite reaction.

Pulleys

A system with a smooth pulley means the tension of the string is the same on both sides of the pulley. You cannot treat a pulley system as a single particle as these particles move in opposite directions.

Example 7: Particles *P* and *Q*, of masses *2m and* 3*m*, are attached to the ends of a light inextensible string. The string passes over a small smooth fixed pulley and the masses hang with the string taut. The system is released from rest.

Find the acceleration of each mass.



For P : $T-2mg=2ma$
For Q : $3mg-T=3ma$

To find acceleration, both equations P and Q should be added together :

$3mg-T+T-2mg=3ma+2ma$

$mg=5ma$

$\frac{1}{5}g=a$

$\frac{1}{5}(9.8)=a$

$a=1.96\text{ ms}^{-2}\quad\approx 2.0\text{ ms}^{-2}\text{ (2 s.f.)}$